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ON HOOKE'S LAW

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Abstract: It is demonstrated that for homogeneous isotropic hyperelastic material, for which Poisson's constant is different from 1/4, Hooke's law cannot hold in the theory of finite deformations. It is then proved that this law is a linear approximation of the constitutive relations.

Key words: continuum mechanics

1. INTRODUCTION

Physical modelling is one of the most popular methods in studying geomechanical processes. The admirers of this method make reference mainly to the paper by Kuznetsov *et al.* (1959) in which the theoretical foundations of this method are explained. One of the assumptions of this method is that Hooke's law is valid for homogeneous isotropic hyperelastic material. In the present paper we shall show that Hooke's law cannot be valid for these materials in the theory of finite deformations if Poisson's constant σ is different from $1/4$.

2. BASIC CONCEPTS[1)]

Body B in R^3 will be described by Cartesian coordinates x^i. The motion of the body is understood to obey the following equation

$$y^i = y^i(x, t) \tag{1}$$

which assigns a position to every point of body B in R^3 at time t. The deformation gradient of the motion will be defined by

$$F^i_k(x, t) = \frac{\partial y^i}{\partial x^k} \tag{2}$$

and the left Cauchy-Green strain tensor by

$$B^{ij}(y, t) = g^{kl} F^i_k F^j_l \tag{3}$$

1) For more details see Leigh (1968).

where $g =$ diag (1,1,1), and for a fixed t, x^i is substituted by y^k which is obtained for this t by inverting Eq. (1). In continuum mechanics constitutive relations are very important. These relations express the dependence of stress tensor T^{ij} on the strain tensor. Leigh (1968) has proved for homogeneous isotropic material

$$\mathsf{T} = \varphi_0 + \varphi_1 \mathsf{B} + \varphi_2 \mathsf{B}^2 \tag{4}$$

where φ_0, φ_1 and φ_2 are functions of invariants I_j, $j = 1, 2, 3$ of B only, namely

$$I_1 = \mathrm{Tr}\mathsf{B} \tag{5}$$

$$I_2 = \tfrac{1}{2}(I_1^2 - \mathrm{Tr}\mathsf{B}^2) \tag{6}$$

$$I_3 = \det\mathsf{B} = I_1 I_2 + \tfrac{1}{3}(\mathrm{Tr}\mathsf{B}^3 - I_1^3) \tag{7}$$

For homogeneous isotropic hyperelastic material it has been proved that function W, called the energy of deformation, exists for which

$$\mathsf{T} = 2\varrho \mathsf{B}\,\frac{\partial W}{\partial \mathsf{B}}\,,$$

where ϱ is the density of material. If we carry out the derivations and apply the continuity equation, we obtain

$$\mathsf{T} = \frac{2\varrho_0}{\sqrt{I_3}}\left[I_3 \frac{\partial W}{\partial I_3} + \left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)\mathsf{B} - \frac{\partial W}{\partial I_2}\,\mathsf{B}^2\right] \tag{8}$$

where ϱ_0 is the density of the undeformed body. If we now compare (4) and (8), we obtain

$$\varphi_0 = 2\varrho_0 \sqrt{I_3}\,\frac{\partial W}{\partial I_3} \tag{9}$$

$$\varphi_1 = \frac{2\varrho_0}{\sqrt{I_3}}\left(\frac{\partial W}{\partial I_1} + I_1\,\frac{\partial W}{\partial I_2}\right) \tag{10}$$

$$\varphi_2 = \frac{-2\varrho_0}{\sqrt{I_3}}\,\frac{\partial W}{\partial I_2} \tag{11}$$

If the material is homogeneous, isotropic and hyperelastic, function W must exist, and relations (9) – (11) have to hold for φ_i. Necessary and sufficient conditions for the existence of a solution to this system of equations are

$$\frac{\partial \varphi_0}{\partial I_1} + I_1\,\frac{\partial \varphi_0}{\partial I_2} - I_3\,\frac{\partial \varphi_1}{\partial I_3} - \tfrac{1}{2}\varphi_1 = 0 \tag{12}$$

$$\frac{\partial \varphi_0}{\partial I_2} + I_3\,\frac{\partial \varphi_2}{\partial I_3} + \tfrac{1}{2}\varphi_2 = 0 \tag{13}$$

$$\frac{\partial \varphi_1}{\partial I_2} + I_1\,\frac{\partial \varphi_2}{\partial I_2} + \frac{\partial \varphi_2}{\partial I_1} = 0 \tag{14}$$

3. HOOKE'S LAW

Hooke's law states that a stress tensor is a linear function of the strain tensor. Assume this to be true. We now introduce a strain tensor $\mathbf{E}$,

$$2\mathbf{E} = \mathbf{B} - \mathbf{1} \,. \tag{15}$$

In this case

$$I_1 = 3 + 2E_1 \,, \tag{16}$$

where $E_1 = \mathrm{Tr}\mathbf{E}$. In this notation Hooke's law takes the form

$$\mathbf{T} = \lambda E_1 + 2\mu\mathbf{E} \tag{17}$$

where λ and μ are Lamé's constants. If strain tensor $\mathbf{B}$ and I_1 from (15) and (16) are substituted for $\mathbf{E}$ and E_i in (17),

$$\mathbf{T} = \frac{\lambda}{2}(I_1 - 3) - \mu + \mu\mathbf{B}$$

Consequently,

$$\varphi_0 = \frac{\lambda}{2}\,(I_1 - 3) - \mu$$

$$\varphi_1 = \mu$$

$$\varphi_2 = 0$$

Integrability conditions (12) – (14) very easily yield

$$\lambda = \mu$$

If we now define Poisson's constant as

$$\sigma = \frac{\lambda}{2(\lambda + \mu)} \tag{19}$$

(Brdička, 1959), it can be easily seen that $\sigma = 1/4$. Now it is evident that for homogeneous isotropic hyperelastic material relations (17) cannot hold unless $\sigma = 1/4$.

It is remarkable that, according to Brdička (1959), we can derive relation (18) from the concept that an elastic body is composed of material points which interact by means of central intermolecular forces. However, experiments have proved that this relation is not valid in general, and that Hooke's law contains two independent constants. A consequence of the above analysis is that, if $\sigma \neq 1/4$, the linear relation between the stress and the strain tensors cannot exist. To explain this apparent contradiction between experiment and theory, we assume that general relation (4) between the stress and strain tensors holds and, in the power series of this relation, we restrict ourselves to the lowest powers in $\mathbf{E}$. Let E_1, E_2 and

E_3 be invariants of tensor $\mathbf{E}$, defined similarly as I_1, I_2 and I_3 for tensor $\mathbf{B}$. Let $\mathbf{T} = \mathbf{0}$ for $\mathbf{B} = \mathbf{1}$. In this case $I_1 = I_2 = 3$ and $I_3 = 1$ or $\mathbf{T} = \mathbf{0}$ for $\mathbf{E} = \mathbf{0}$ and $E_1 = E_2 = E_3 = 0$. We develop equation (4) in the neighbourhood of this point into a power series and we restrict ourselves to the quadratic term in $\mathbf{E}$. Longer but straightforward calculations yield

$$\begin{aligned}\mathbf{T} = \Sigma + 2(\Sigma_{,1} + 2\Sigma_{,2} + \Sigma_{,3})\, E_1 + 2\psi\mathbf{E} + 4(\Sigma_{,2} + \Sigma_{,3})\, E_2 + \\ + 2(\Sigma_{,11} + 4\Sigma_{,12} + 2\Sigma_{,13} + \Sigma_{,22} + 4\Sigma_{,23} + \Sigma_{,33})\, E_1^2 + \\ + 4(\psi_{,1} + 2\psi_{,2} + \psi_{,3})\, E_1\mathbf{E} + 4\varphi_2\mathbf{E}^2 + o(||\mathbf{E}||^2)\end{aligned} \tag{20}$$

where

$$\Sigma = \varphi_0 + \varphi_1 + \varphi_2$$

$$\psi = \varphi_1 + 2\varphi_2$$

and $\Sigma_{,\alpha}$ or $\psi_{,\alpha}$ denote differentiation with respect to I_α, etc., and all values of Σ, ψ, φ_2 and their derivatives are taken at point (3,3,1). Since $\mathbf{T} = \mathbf{0}$ for $\mathbf{E} = \mathbf{0}$, $\Sigma = 0$, and thus

$$\mathbf{T} = \lambda E_1 + 2\mu\mathbf{E} + \alpha_1 E_1^2 + \alpha_2 E_2 + \alpha_3 E_1\mathbf{E} + \alpha_4 E^2 + o\,(||\mathbf{E}||^2) \tag{21}$$

where λ, μ, α_i are constants. It is easy to satisfy oneself that the conditions of integrability impose no constraints on constants λ and μ in (21). Therefore, the linear part of the power series contains two independent constants and, if we restrict ourselves to the linear approximation, we obtain Hooke's law.

Moreover, experiments indicate that in some neighbourhood of point $\mathbf{T} = \mathbf{0}$ the relation between the stress and strain tensors is invertible. According to the implicit function theorem (Jarník, 1976)

$$\det\left(\frac{\partial T^{ij}}{\partial B^{kl}}\right) \neq 0 \quad \text{at point} \quad \mathbf{B} = \mathbf{1}$$

Some easy algebra yields

$$\psi^2[\psi + 3(\Sigma_{,1} + 2\Sigma_{,2} + \Sigma_{,3})] \neq 0$$

for this determinant. This nonequation holds iff $\psi \neq 0$ and $\psi + 3(\Sigma_{,1} + 2\Sigma_{,2} + \Sigma_{,3}) \neq$ $\neq 0$. It is evident from (20) and (21) that this condition is equivalent to

$$\mu \neq 0 \quad \text{and} \quad 3\lambda \neq 0\,.$$

If we extend our deliberations also to thermodynamic processes, it is possible to prove (Leigh, 1968) that for λ and μ the inequalities

$$\mu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0\,,$$

must hold, respectively. This is, for example, what Brdička (1959) claims. In the hyperelastic case, the above-mentioned relations lead to the following relations for W

$$W_{,1} + 2W_{,2} + W_{,3} = 0 \tag{22}$$

and

$$0 < W_{,1} - W_{,3} < 3(W_{,11} + 4W_{,12} + 2W_{,13} + 4W_{,22} + 4W_{,23} + W_{,33}) \tag{23}$$

where all derivatives of W are again taken at point (3,3,1). Conversely, for homogeneous isotropic hyperelastic material, any function W for which conditions (22) and (23) are satisfied defines a constitutive equation with the property that for some neighbourhood of point $\mathbf{T} = \mathbf{0}$ the relation between the stress and strain tensors is bijective.

4. CONCLUDING REMARKS

The above analysis of the constitutive relation between the stress and strain tensors for homogeneous isotropic hyperelastic materials clearly indicates that, if Poisson's constant is different from 1/4, the assumption of the validity of Hooke's law implicitly made by Kuznetsov *et al.* (1959), which is one of the main assumptions of physical modelling, is a mere linearization of the constitutive equations. Consequently, not even in the case of the simplest materials can it be assumed that a model is equivalent to the original, even if we conform to all criteria of similarity, given in Kuznetsov's *et al.* (1959) paper.

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K HOOKOVU ZÁKONU

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V článku je dokázáno, že pro homogenní izotropní hyperelastický materiál, pro nějž je Poissonova konstanta různá od 1/4, nemůže v teorii konečných deformací platit Hookův zákon. Dále se ukazuje, že tento zákon je linerání aproximací konstitutivních vztahů.

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