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ON THE RELATION BETWEEN THE LINEARIZED THEORY OF FINITE DEFORMATIONS AND THE THEORY OF SMALL DEFORMATIONS

ONDŘEJ NAVRÁTIL

Institute of Geotechnics, Czechoslovak Academy of Sciences,
V Holešovičkách 41, 182 09 Prague, Czechoslovakia

Abstract: Linearized equations of motion for finite deformations of an elastic body, previously deformed by the action of external forces, are derived. It is shown that these equations describe small deformations iff the stress tensor for rest loading is zero.

Key words: continuum mechanics

1. INTRODUCTION

Mathematical modelling of geomechanical processes is based on the assumption that the deforming medium behaves like a continuum. Therefore, in geomechanics we have to concern ourselves with continuum mechanics. At present, the method of mathematical modelling is mainly limited to the assumption of small deformations (Procházka, 1990). We are not sure that this assumption is always justified. This is the reason why we want to discuss the question as to when the equations of motion for finite deformations may be substituted, at least infinitesimally, by the well-known equations for small deformations. In this paper we shall only restrict ourselves to the case of elastic material, assuming that the density of volume forces per unit mass is constant in the whole body.

2. BASIC CONCEPTS AND EQUATIONS OF CONTINUUM MECHANICS[1)]

A body is a connected set in R^3 on which it is possible to define the structure of the C^∞ manifold which has dimension 3. We shall denote the boundary of body B by δB. We assume that the mapping $\chi : B \to \mathrm{R}^3$ which projects B as a whole on R^3, and the inverse mapping $\chi^{-1} : \chi(B) \to B$ exist. Any of these functions will be called the configuration of body B. The set of configurations $\varphi_t, t \in \mathrm{R}$ such that the function $\chi_t = \varphi_t \circ \chi^{-1} : \mathrm{R} \times \chi(B) \to \mathrm{R}^3$ is C^∞ differentiable for any configuration χ will be referred to as the motion of body B.

[1)] For more details see Leigh (1968).

To throw some light on these abstract terms we note that body B is an abstract topological subset of R^3. The structure of the C^∞ manifold allows us to introduce local coordinates into this set, and the existence of the configuration allows us to define these coordinates globally (of course, this assumption is not necessary). Motion φ_t of the body in some configuration χ assigns, to any $\xi \in \chi(B)$, a curve $\xi(t)$, which is the motion of a point of body B.

The motion of body B in coordinates will be denoted by $x^i = x^i(\xi, t)$. Next we shall define the velocity of motion: $V^i(\xi, t) = (\partial x^i/\partial t)(\xi, t)$. Since function ξ_t has an inverse function for every t, we can express ξ as a function of x and $t, \xi(x,t)$ and define the velocity: $v^i(x,t) = V^i(\xi(x,t),t)$. Defining the acceleration by $A^i(\xi, t) = (\partial V^i/\partial t)(\xi, t)$, we have $a^i(x,t) = A^i(\xi(x,t),\, t) = (\partial v^i/\partial t) + (\partial v^i/\partial x^j)\ v^j$. Next we put $\dot{x}^i$ for the velocity and $\ddot{x}^i$ for the acceleration regardless of whether it is a function of ξ or x. Now we shall concern ourselves with the description of the deformation of body B. Let the motion of the body be described in the configuration χ by the equations $x^i = x^i(x,t)$. In this case the function

$$X^i_\alpha(\xi, t) = \frac{\partial x^i}{\partial \xi^\alpha}\,(\xi, t) \tag{1}$$

is called the deformation gradient of motion x^i. $X = \det X^i_\alpha \neq 0$ holds for the deformation gradient of motion. Consequently, we can assume, as we do further on, that $X > 0$ for every t. It is convenient to introduce the right Cauchy-Green strain tensor by the relation

$$C_{\alpha\beta}(\xi, t) = g_{kl} X^k_\alpha\, X^l_\beta \tag{2}$$

where g_{kl} is a unit tensor and $g =$diag (1, 1, 1). To render the definitions for finite and small deformations uniform, we shall denote the strain tensor by

$$c_{\alpha\beta} = \tfrac{1}{2}(C_{\alpha\beta} - g_{\alpha\beta}) \tag{3}$$

We assume that function ϱ, the density, is defined on body B. The equation of continuity holds for this function, namely

$$\dot{\varrho} + \varrho\,\frac{\partial v^i}{\partial x^i} = 0 \tag{4}$$

Equation (4) can be expressed in integral form

$$\varrho(x \tag{5}$$

where $\hat{\varrho}$ is the density in configuration ξ and ϱ the density at point $x = x(\xi, t)$.

External forces act on the body. Their density per unit mass is b^i. Let b^i be constant in the whole body. The surface forces in the body are determined by the symmetric stress tensor $T^{ij}(x,t)$. If σ is a regular orientable surface inside body B, the force acting on this surface is $f^i = \int_\sigma T^{ij} n_j \,\mathrm{d}S$, where n_j is the unit vector of the normal to surface σ. We now can express the equation of motion in this notation as

$$\varrho(x,t)\,\ddot{x}^i = \varrho(x,t)\,b^i + \frac{\partial}{\partial x^j}\,T^{ij}(x,t) \tag{6}$$

Equations (4) and (6) determine the motion of the body if we choose the instantaneous state of the body as the reference configuration for every t. If we choose any fixed configuration ξ as the reference configuration, the equations of motion have the following form

$$\hat{\varrho}(\xi)\,\ddot{x}^i(\xi,t) = \hat{\varrho}(\xi)\,b^i + \frac{\partial}{\partial \xi^\alpha}\,S^{i\alpha}(\xi,t) \tag{7}$$

where $S^{i\alpha}(\xi,t)$ is the Piola stress tensor, defined by the relation

$$S^{i\alpha}(\xi,t) = XT^{ij}(x(\xi,t),\ t)\ (X^{-1})^\alpha_j\ (\xi,t) \tag{8}$$

The equations of motion and the equation of continuity still do not describe the motion of the body uniquely. It is necessary to define the constitutive equations which give the relations between the stress and the strain tensor. These equations depend, of course, on the material of the body. In the theory of elasticity we assume that the stress tensor at point x is a function of the deformation gradient of motion at this point, i.e. with regard to the reference configuration

$$T^{ij}(x,t) = F^{ij}(X^k_\alpha(\xi,t),\xi) \tag{9}$$

where ξ is the solution to the equation $x^i = \chi^i_t(\xi)$. According to the principle of material objectivity (Leigh, 1968), which states that the equations describing physical processes are independent of the observer, it follows that

$$Q^i_k Q^j_l F^{kl}(X^r_\alpha,\xi) = F^{ij}(Q^r_s X^s_\alpha,\xi) \tag{10}$$

for any orthogonal Q, i.e. for any Q for which

$$g_{kl} Q^k_i Q^l_j = g_{ij}$$

From condition (10) it is possible to derive that

$$T^{ij}(x,t) = X^{-1} X^i_\alpha X^j_\beta R^{\alpha\beta}(c_{\varrho\sigma},\xi) \tag{11}$$

must hold. The other relations for the stress tensor can be derived if we assume the existence of an isotropy group, i. e. a group of transformations of the reference configuration for which

$$F^{ij}(X^r_\alpha,\xi) = F^{ij}(X^r_\beta G^\beta_\alpha,\xi)$$

where det $G = 1$. In view of thermodynamics we can prove that

$$R^{\alpha\beta}(c_{\varrho\sigma}, \xi) = \frac{\partial W}{\partial c_{\alpha\beta}} (c_{\varrho\sigma}, \xi) \tag{12}$$

where $(\partial^2 W / \partial c_{\alpha\beta} \partial c_{\varrho\sigma})$ is the positive definite mapping $S^2R^3 \times S^2R^3$ to R. It is interesting to note that if the reference configuration $\bar{\xi} = \phi^\alpha(\xi)$ is changed, we obtain

$$\bar{R}_{\alpha\beta}(\bar{c}, \bar{\xi}) = K^{-1} K^{\alpha\varrho} K^{\beta\sigma} R^{\varrho\sigma}(K^\gamma_\pi K^\delta_\omega c_{\gamma\delta} + p_{\pi\omega}, \xi) \tag{13}$$

where $K^\alpha_\beta = (\partial\bar{\xi}^\alpha / \partial\xi^\beta)$ and $p_{\alpha\beta} = \frac{1}{2}(g_{\varrho\sigma} K^\varrho_\alpha K^\sigma_\beta - g_{\alpha\beta})$; therefore, the form of equation (11) is independent of the choice of the reference configuration. However, this is also one of the requirements imposed on this theory.

In the theory of small deformations we define the tensor of small deformations by the formula

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial x_i}{\partial \xi^j} + \frac{\partial x_j}{\partial \xi^i}\right) - g_{ij} \tag{14}$$

and assume that the stress tensor is a function of ε_{ij}, i.e.

$$T^{ij}(x,t) = H^{ij}(\varepsilon_{kl}, \xi) \tag{15}$$

The tensor of small deformations (14) corresponds to strain tensor (3) in which we omit the terms $g_{kl}[(\partial u^k / \partial\xi^\alpha)\ (\partial u^l / \partial\xi^\beta)]$, where $u^k = x^k - \xi^k$. This omission of quadratic terms is explained by the assumption that the deformations are small. This corresponds to the small deformations from the reference configuration. Therefore, these deformations are called small. If we make this omission also in equation (8), we get equation of motion

$$\hat{\varrho}(\xi)\ddot{x}^i(\xi, t) = \hat{\varrho}(\xi)\, b^i + \frac{\partial}{\partial \xi^j} H^{ij}(\varepsilon, \xi) \tag{16}$$

which is known from the classical theory of elasticity (Brdička, 1959). If we now choose another reference configuration $\bar{\xi}$, we get the relation

$$\varepsilon_{ij} = M_{ij} + \frac{1}{2}\left(K^k_i \frac{\partial u_j}{\partial \bar{\xi}^k} + K^k_j \frac{\partial u_i}{\partial \bar{\xi}^k}\right)$$

where $M_{ij} = \frac{1}{2}(K^i_j + K^j_i) - g_{ij}$. It is evident that in this case we cannot express the tensor of small deformations in terms of the original tensor of small deformations and functions K^i_j. Therefore, in the case of these transformations the tensor of small deformations does not behave as a tensor quantity, and the form of the equation of motion (16) depends on the choice of the reference configuration. Assuming that functions K^i_j differ only a little from unity and omitting the quadratic terms $(K^i_j - \delta^i_j)\ (\partial u_k / \partial\bar{\xi}^i)$, we obtain the transformation relation

$$\varepsilon_{ij} = M_{ij} + \bar{\varepsilon}_{ij}$$

Nevertheless, these are only small changes of the reference configuration. Consequently,
can only hold in a particular configuration, but not in general. The existence of this configuration is essential for the theory of small deformations. In other configurations the stress tensor depends not only on the tensor of small deformations but generally on the deformation gradient of motion.

3. FORMULATION OF THE PROBLEM

Let us formulate now one problem studied in the theory of elasticity.
be composed of the material for which constitutive equations are known with respect to any configuration ξ and have the form of (11). Let the body be in equilibrium at time t_0 and be projected onto R^3 by means of configuration $\chi : B \to \mathrm{R}^3$; $\chi^i(p) = x^i$, where $p \in B$. Let stress tensor $T^{ij}(x)$, density $\varrho(x)$ and density of external forces b^i for any $x \in \chi(B)$ be known in this configuration. The boundary of body B will be divided into two disjunctive subsets $\delta_1 B$ and $\delta_2 B$ where $\delta B = \delta_1 B \cap \delta_2 B$. Now we shall apply the external forces to the boundary of body B. Consequently, body will move. This motion will be described by function $y^i = y^i(x,t)$. We know the distribution the of external surface forces on $\delta_1 B$ in configuration y and values $y^i(x,t)$ on $\delta_2 B$. The task is to find $y^i(x,t)$ and $T^{ij}(y,t)$ within and on the boundary of body B. It is not simple to solve this problem. We now have three, in general different, configurations and we must find transformations between them. The first problem is to find constitutive equations with regard to the reference configuration x^i, i.e. to solve the equations

$$T^{ij}(x) = X^{-1} X^i_\alpha X^j_\beta R^{\alpha\beta}(c_{\varrho\sigma})$$

If posssible, we determine the deformation gradients X^i_α from these equations. Let the solutions to these equations exist. This is, of course, already a condition being imposed on stress tensor $T^{ij}(x)$. The next condition, which must be fulfilled by stress tensor T^{ij}, is the equation of equilibrium

$$\varrho(x)\, b^i + \frac{\partial T^{ij}}{\partial x^j} = 0 \tag{17}$$

If also the external forces, which act on the boundary of body B, are known, then

$$T^{ij}(x)\, n_j(x) = \sigma^i(x) \tag{18}$$

must hold on this boundary, where n_j is the a unit vector of the normal external to boundary δB. When all these conditions are fulfilled we must solve the system of partial differential equations of the second order

$$\varrho(x)\, \ddot{y}^i(x,t) = \varrho(x)\, b^i + \frac{\partial}{\partial x^j}\, \left(p^i_k R^{kj}(E_{rs}, x)\right) \tag{19}$$

where $p^i_j = \partial y^i/\partial x^j$ is the deformation gradient of the motion y^i with regard to configuration $x, E_{ij} = \frac{1}{2}($
deformation gradient, and $J = \det p > 0$. The solution to these equations must satisfy the boundary conditions, which we can formulate in the following way: Let part $\delta_1 B$ of the boundary of body B be described in configuration x by the equation $\varphi(x) = 0$. During motion this surface changes into another surface, described by the relation $\bar{\varphi}(y,t) = \bar{\varphi}(y(x,t),\ t) = \varphi(x) = 0$. The solution on this surface must conform to the condition

$$T^{ij}(y)N_j(y) = J^{-1}p^i_k p^j_l R^{kl}(E$$

where N_j is the unit normal to surface $\bar{\varphi}(y,t) = 0$ and $\bar{\sigma}^i(y,t)$ is the known density of external surface forces on $\delta_1 B$. For part $\delta_2 B$ of boundary δB, on which we assume that functions $y^i($,
hold, where $\psi(x) = 0$ describes boundary $\delta_2 B$ before the deformation. We can see that equations (19) and boundary conditions (20) are nonlinear. It is, therefore, very difficult to solve these equations.

In solving this problem in the theory of small deformations, we must first ascertain that the deformations which correspond to the initial stress tensor are small with regard to the configuration for which the constitutive equations are known, and that formula (15) holds for these equations. These are the conditions for which we have derived the equations of motion (16). In the same approximation, which we have already made, we can express the boundary condition as

$$H^{ij}(\varepsilon,\xi)\, n_j(\xi) = \bar{\sigma}^i(\xi) \tag{21}$$

where n_j is the unit normal external to the boundary in reference configuration ξ. Thus, after all these approximations we obtain the equations of the classical theory of elasticity (

4. LINEARIZED EQUATIONS OF FINITE ELASTICITY

In this section, we shall derive the linearized equations corresponding to the equations of motion (19) and boundary condition (20). There is some analogy here to the equations which describe small deformations. These equations describe small deformations with regard to the previously known finite deformation for which the stress tensor is $T^{ij}(x)$.

Let the stress tensor satisfy equation (17) and boundary condition (18), and constitutive equations be given with regard to the configuration x by the following relations

$$T^{ij}(y,t) = J^{-1}p^i_k p^j_l R^{kl}(E_{rs}, \tag{22}$$

The equation of motion (19) and boundary conditions (20) hold. We shall seek the solution to this equation of motion in the form of $y^i = x^i + \tau\eta^i($
substituting into (19) and (20), we obtain

$$\varrho(x)\ddot{\eta}^i(x,t) = \varrho(x)\, b^i + \frac{\partial}{\partial x^j}\left[(\delta^i_k + \tau\eta^i_k)\, R^{kj}\left(\tau e_{rs} + \frac{\tau^2}{2}\, g_{tv}\eta^t_r\, \eta^v_s, x\right)\right]$$

and

$$\bar{\sigma}^i(x + \tau\eta) = \sigma^i(x) + \tau\Sigma^i(x,t) + o(\tau)$$

where $\eta^i_k = (\partial\eta^i/\partial x^k)$ and $e_{ij} = \frac{1}{2}(g_{ir}\eta^r_j + g_{jr}\eta^r_i)$ and $\Sigma^i(x,t)$ is the linear part of the change in the density of the surface forces. We shall develop the equations of motion and boundary condition into a power series restricting ourselves to the linear part only. After some easy algebra involving Eqs. (19) and (20) we get

$$\varrho(x)\ddot{\eta}^i(x,t) = \frac{\partial}{\partial x^j}\left[T^{jk}(x)\eta^i_k + \frac{\partial R^{ij}}{\partial c_{rs}}(0,x)\, e_{rs}\right] \tag{23}$$

Introducing $\omega_{rs} = \frac{1}{2}(g_{ri}\eta^i_s - g_{si}\eta^i_r)$ we can put

$$\varrho(x)\ddot{\eta}^i(x,t) = \frac{\partial}{\partial x^j}\left[\left(\frac{\partial R^{ij}}{\partial c_{rs}}(0,x) + T^{js}(x)\, g^{ir}\right) e_{rs} + T^{js}(x)\, g^{ir}\omega_{rs}\right] \tag{24}$$

In this approximation the boundary condition has the following form

$$\left(\frac{\partial R^{ij}}{\partial c_{rs}}\, e_{rs} + T^{rj}\eta^i_r\right) n_j = \sigma^i(e^r_r - e^{rs}n_r n_s) + \Sigma^i \tag{25}$$

which holds along boundary $\delta_1 B$ of body B in configuration x. It is interesting to note that, if $T^{ij} = 0, b^i = 0$ and $\sigma^i = 0$ and, in configuration x, the constitutive equations take the following form:

$$T^{ij}(y) = \frac{\partial R^{ij}}{\partial c_{rs}}(0,x)\, e_{rs}$$

We have obtained the same equations as for the small deformations. But it is not true that for these conditions and in this approximation we should obtain the theory of small deformations because in our case the constitutive equations are linear in e_{rs} whereas this need not be true in the theory of small deformations. Equations (24) and (25) do not depend only on terms e_{rs} as with small deformations but also on variables ω_{rs} which determine infinitesimal rotations. We shall now introduce

$$A^{ij,rs} = \tfrac{1}{2}(T^{jr}g^{is} + T^{js}g^{ir}) + \frac{\partial R^{ij}}{\partial c_{rs}} \tag{26}$$

$$Z^{ij,rs} = \tfrac{1}{2}(T^{js}g^{ir} - T^{jr}g^{is}) \tag{27}$$

In this notation we can express (24) and (25) in the form

$$\varrho\ddot{\eta}^i = \frac{\partial}{\partial x^j}\,(A^{ij,rs}e_{rs} + Z^{ij,rs}\omega_{rs}) \tag{28}$$

and

$$(A^{ij,rs}e_{rs} + Z^{ij,rs}\omega_r s)\, n_j = \sigma^i(e^r_r - e^{rs}n_r n_s) + \Sigma^i \tag{29}$$

We can see that $A^{ij,rs}$ and $Z^{ij,rs}$ determine the equations of motion and the boundary conditions. However, these quantities cannot be arbitrary. Equations (26) and (27) indicate that

$$A^{ij,rs} = A^{ij,sr} \tag{30}$$

and

$$Z^{ij,rs} = -Z^{ij,sr} \tag{31}$$

But $A^{ij,rs}$ and $Z^{ij,rs}$ must satisfy more relations. Let

$$B^{ij,rs} = A^{ij,rs} - A^{ji,rs} \tag{32}$$

Of course $B^{ij,rs} = B^{ij,sr} = -B^{ji,rs}$. Let

$$D^{j,r} = B^{ij,rs} g_{is} \tag{33}$$

For a solution to (26) to exist in $A^{ij,rs}, D^{j,r}$ and $B^{ij,rs}$ must satisfy the following relations

$$D^{j,r} = D^{r,j} \tag{34}$$

$$D^{j,r} g_{jr} = 0 \tag{35}$$

and

$$B^{ij,rs} = \tfrac{1}{3}(D^{j,r}g^{is} + D^{j,s}g^{ir} - D^{i,r}g^{js} - D^{i,s}g^{jr}) \tag{36}$$

The symmetric part $A^{ij,rs} + A^{ji,rs}$ and $D^{j,r}$, for which (34) and (35) hold true, determine $A^{ij,rs}$ uniquely. Therefore, A is defined by 36 values of its symmetric part and by five values of D, i.e. by 41 parameters. For the hyperelastic case (12) holds and, therefore, the symmetric part of A is invariant to the change $(ij) \leftrightarrow (rs)$ and, consequently, the symmetric part is only defined by 21 parameters.

We shall now study the conditions under which (27) has a solution. Let

$$G_t^{ij} = \tfrac{1}{2}\varepsilon_{rst} Z^{ij,rs} \tag{37}$$

where ε_{rst} is the Levi-Cività tensor. In this case (27) has a solution iff

$$gij G_s^{ij} = 0 \tag{38}$$

and

$$g_{ri} G_s^{ij} + g_{si} G_r^{ij} = 0 \tag{39}$$

These equations have solutions iff T^{kl}, for which

$$T^{ij} = T^{ji} \tag{40}$$

and

$$G_r^{ij} = \tfrac{1}{2} g^{is} \varepsilon_{rst} T^{jr} \tag{41}$$

exists. It can be easily proved that T^{ij} is the initial stress tensor. Therefore Z is defined by six parameters T and, conversely, Z defines T uniquely. The solvability of the system of equations (26) and (27) implies the relation between $D^{j,r}$ and T^{jr}:

$$D^{j,r} = 3T^{jr} - T^s_s g^{jr} \tag{42}$$

Therefore, the solution to systems (26) and (27) is defined by 36 values of the symmetric part A (in the hyperelastic case by 21) and by six values of T^{ij}. This is not surprising because in linear terms we can put

$$T^{ij}(x + \tau\eta) = T^{ij}(x) + \frac{\tau}{2}[(A^{ij,rs} + A^{ji,rs}) + (Z^{ij,rs} + Z^{ji,rs})\omega_{rs} - T^{ij}(x)\, e^r_r]$$

As opposed to Hooke's law (Brdička 1959), stress tensor $T^{ij}(x)$ appears here. But we can express the terms by which this relation differs from Hooke's law with the aid of the known stress tensor T. It is interesting that the antisymmetric part of A does not depend on the trace of the stress tensor and only is a function of the deviator of this tensor.

5. CONCLUSION

From the above analysis it is evident that the linearized equations of motion for finite deformations are reduced to the equations for small deformations iff the tensor of initial stress $T^{ij}(x)$ is equal to zero. But this is never the case in geotechnics because the forces of gravity always exist. Therefore, not even in a linearized case can we solve the problem defined in Sect. 2 of this paper by means of the theory of small deformations. This is a question of how much the solution to this problem in the linearized theory of finite deformations differs from its solution in the theory of small deformations.

REFERENCES

Brdička M. (1959). *Continuum mechanics,* SNTL, Prague (in Czech).

Leigh, D.C. (1968). *Nonlinear continuum mechanics,* McGraw-Hill, New York, 240 pp.

Procházka, P. (1990). *Optimization and contact problems in geotechnics,* Dr.Sc. Thesis, IGt, CSAS, Prague.

O VZTAHU MEZI LINEARIZOVANOU TEORIÍ KONEČNÝCH DEFORMACÍ A TEORIÍ MALÝCH DEFORMACÍ

Ondřej Navrátil

V článku jsou odvozeny linearizované pohybové rovnice pro elastické těleso, které bylo předem deformováno působením vnějších sil. Je ukázáno, že tyto rovnice jsou rovnice popisující malé deformace, právě když je tenzor napětí pro počáteční zatížení nulový.

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