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THE BEM APPLIED TO OPTIMIZATION AND CONTACT PROBLEMS IN GEOTECHNICS

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Abstract: In the paper the application of the Boundary Element Method to selected geotechnical problems is presented. Special attention is devoted to the analysis of the stress state in rock due to openings with or without stiffeners (tunnel wall, lining, etc.). The formulation is proposed as a coupling of contact and optimization problems. A typical example demonstrates the behaviour of the model.

Key words: optimal shape design; contact problem; Uzawa's algorithm; openings in rock; stress state in rock

1. INTRODUCTION

In this paper we will deal with the application of the boundary element method (BEM) (see Banerjee and Butterfield, 1981; Brebbia and Walker, 1980; Brož and Procházka, 1987; Crouch and Starfield, 1983) to the solution of some geotechnical problems for which the BEM is extraordinarily advantageous. Among others we can name problems requiring the underground continuum to be expressed as a three-dimensional one (e.g. a halfspace problem). The BEM reduces the dimension of the problem by one and makes it possible to include a point in infinity into the domain. Another well-suited use of the method is its application to optimization (see Céa, 1971; Duvant and Lions; 1972; Haslinger and Neittaanmaki; 1988; Horák and Procházka, 1987) and/or contact problems which cover non-linearities at boundaries of the domains only (see Brož and Procházka, 1985). Then, as opposed to the FEM (finite element method) (see Janovský and Procházka, 1980) it suffices to study the influence of the boundary elements on the solution.

In this paper we will concentrate on two-dimensional problems only, while generalisation to three dimensions has a formal character.

We start with the static equations:

$$(\lambda + \mu)\frac{\partial}{\partial x_i}\operatorname{div}\boldsymbol{u} + \mu \Delta u_i + b_i + \frac{\partial \sigma^0_{ij}}{\partial x_j} = 0, \qquad i, j = 1, \dots, 2 \tag{1.1}$$

where

$$\operatorname{div}\boldsymbol{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

and $\boldsymbol{u} = (u_1, u_2)$ is the vector of displacement, b_1, b_2 are components of the vector of volume weight, σ^0_{ij} components of the tensor of initial stress.

These equations will be solved in coordinate system Ox_1x_2 on domain Ω:

$$\Omega = \{(x_1, x_2) \in \mathbb{R}_2; \quad x_2 < 0, \quad x_1 \in (-\infty, \infty)\} \tag{1.2}$$

which is a halfspace with boundary Γ being the x_1-axis:

$$\Gamma = \{(x_1, x_2) \in \mathbb{R}_2; \quad x_2 = 0, \quad x_1 \in (-\infty, \infty)\} \tag{1.3}$$

In the sense of the BEM it is possible to re-formulate Eq. (1.3) into an equivalent form (see Brož and Procházka, 1987):

$$c_{kl}u_l(\xi) = (\varepsilon^*_{ijk}, \sigma^0_{ij}) + [p^*_{ik}, u_i] - [u^*_{ik}, p_i] - (u^*_{ik}, X_i) \tag{1.4}$$

where $c_{kl} = \delta_{kl}$ for $\xi \in \Omega, c_{kl} = \frac{1}{2}\delta_{kl}$ for $\xi \in$ smooth boundary of Ω, ε is the tensor of deformation, $\boldsymbol{p} = (p_1, p_2)$ are the tractions, and the quantities marked with asterisks are the appropriate source functions.

Since points in infinity have to be involved into domain, instead of the fundamental solution of the adjoint equation, the *Green function* is often used. It obeys the equation:

$$(\lambda + \mu)\frac{\partial u^*_{kj}}{\partial x_i \partial x_j} + \mu \Delta u^*_{ki} + \delta^*_k = 0, \quad i, j, k = 1, \dots, 2 \tag{1.5}$$

where δ^*_k is a point load, i.e. vector $(\delta_{1k}, \delta_{2k})$ δ, δ_{ik} being the Kronecker symbol, and δ the Dirac function. Moreover, the *Green function* $\boldsymbol{u}^*$ fulfils the condition that tractions are equal to zero on boundary Γ. Such a function was derived by Melan (1932) and was used for the purpose of the BEM in Brož and Procházka (1987, pp. 112–113).

The above-mentioned conditions lead to the following equation, which substitute for (1.5):

$$c_{kl}u_l(\xi) = (\varepsilon^*_{ijk}, \sigma^0_{ij}) - (u^*_{ik}, b_i) \tag{1.6}$$

On the other hand, on the halfspace (ν is the Poisson coefficient)

$$\begin{gathered} u^0_1 = \frac{1 - 2\nu}{4\nu(1 - 2\nu)} X_1 x_1^2 + \text{const.}, \quad u^0_2 = 0 \\ \sigma^0_{11} = -X_1 x_1, \quad \sigma^0_{12} = 0, \quad \sigma^0_{22} = -\frac{\nu}{1 - \nu} X_1 x_1 \end{gathered} \tag{1.7}$$

A combination of (1.6) and (1.7) with apropriate conditions describe the stress state induced by the volume weight.

2. INFLUENCE OF OPENINGS IN A HALFSPACE

Suppose that due to mining activities a cavern or more caverns are open. Denote these caverns by Ω_e and their boundaries by Γ_e. For the sake of simplicity let us consider neither volume weight nor the initial stress. Then

$$c_{kl}u_l(\xi) = \sum_{e=1}^{m}\left(\int_{\Gamma_e} p^*_{ik}(x,\xi)\,u_i(x)\,\mathrm{d}\Gamma_e - \int_{\Gamma_e} u^*_{ik}(x,\xi)\,p_i(x)\,\mathrm{d}\Gamma_e\right) \tag{2.1}$$

Denote $p^0_i(x) = \sigma^0_{ii}(x)$, $x \in \Gamma_e$. Assuming that the boundaries are smooth enough, point ξ may be localized at Γ_e (this point will run along boundary Γ_e) to obtain:

$$\tfrac{1}{2}u_k(\xi) = \sum_{e=1}^{m}\left(\int_{\Gamma_e} p^*_{ik}(x,\xi)\,u_i(x)\,\mathrm{d}\Gamma_e - \int_{\Gamma_e} u^*_{ik}(x,\xi)\,p^0_i(x)\,\mathrm{d}\Gamma_e\right) \tag{2.2}$$

The integral equation (2.2) can be solved by discretizing boundary Γ_e into boundary elements to get the displacements on Γ_e with respect to $\Omega - \Omega_e$ induced by the openings. Now, tractions p^0_i on Γ_e are given and the displacements due to the openings are known at Γ_e from the system (2.2). Under substitution $c_{kl} = \delta_{kl}$ system (2.1) yields the displacements on $\Omega - \Omega_e$. The stress state on $\Omega - \Omega_e$, induced by the openings, and p^0_i can be obtained from a physical law (e.g. Hooke's law) . Denote the components of this tensor by σ^1. For the components of real stress σ_{ij} it holds

$$\sigma_{ij} = \sigma^0_{ij} - \sigma^1_{ij} \tag{2.3}$$

so that a superposition of the state under loading of volume weight of the whole halfspace (see Eq. (1.7)) and the state under loading of boundaries Γ_e so that the tractions along Γ_e, in view of condition (2.3), vanish.

3. INFLUENCE OF STIFFENED OPENINGS ON HALFSPACE

In practical cases openings are sometimes stiffened by linings or tunnel walls. If we proceed via the solution of Eq. (1.1) for a halfspace, it is obvious that the volume weight will again present difficulties. Therefore, we will present a procedure originating from the superposition of the stress state induced by volume weight and the state after opening and stiffening of the tunnel. It is necessary to specify the second state. We will suppose (the procedure is theoretical but can be used in some special cases) that, first of all, the stiffening is carried out and the stiffened tunnel is then opened (the rock removed).

Let us denote the conjunction of Γ_e by Γ_0. The quantities denoted by upper index 1 are related to $\Omega - \Omega_e$ and those by upper index 2 are related to Ω_e. For points ξ on contact Γ_0 it holds

$$\tfrac{1}{2}\,{}^1u_k(\xi) = \sum_{e=1}^{m}\left(\int_{\Gamma_e} {}^1p^*_{ik}(x,\xi)\,{}^1u_i(x)\,\mathrm{d}\Gamma_0 - \right.$$
$$\left. - \int_{\Gamma_e} {}^1u^*_{ik}(x,\xi)\,{}^1p_i(x)\,\mathrm{d}\Gamma_0\right) \tag{3.1}$$

$$\tfrac{1}{2}\,{}^{2}u_k(\xi) = \sum_{e=1}^{m} \left(\int_{\Gamma_e} {}^{2}p^*_{ik}(x,\xi)\,{}^{2}u_i(x)\,\mathrm{d}\Gamma_0 - \right.$$

$$\left. - \int_{\Gamma_e} {}^{2}u^*_{ik}(x,\xi)\,{}^{2}p_i(x)\,\mathrm{d}\Gamma_0 \right) \tag{3.2}$$

Eqs. (3.1) and (3.2) generate a linear system of four integral equations (Fredholm equations of the second kind) in eight unknown functions. It implies that some additional conditions have to be formulated. In our case these are the interface conditions:

$${}^{1}u_i(\xi) = {}^{2}u_i(\xi), \quad \xi \in \Gamma_0 \tag{3.3}$$

$${}^{1}p_i(\xi) + {}^{2}p_i(\xi) = 0, \quad \xi \in \Gamma_0 \tag{3.4}$$

System (3.1) to (3.4) can be solved by the BEM considering the linear states. The displacements and tractions along the boundary Γ_0 are obtained and, from (2.1) at $c_{kl} = \delta_{kl}$, also the displacements on both the subdomains. Following Hooke's law the stresses induced by this process of construction will be at disposal.

So far, the elastic states of material have only been considered. Also, we have not employed the initial stress σ^0_{ij}. We have modelled (for the reason mentioned above) its effect in another way. We will briefly mention a possible way of mathematical modelling the non-linear behaviour (plasticity) of the material. This is similar to the procedure described in Brož and Procházka (1987, Chap. 5.3).

Since the BEM does not permit the change of material constants, the non-linear behaviour has to be expressed by virtue of an incremental method, and the necessary change of quantitites is delivered from the initial stresses.

Let us consider the previous problem under the assumption of non-linear behaviour of material. With respect to Eq. (2.3), instead of (3.3) and (3.4) we get

$$\tfrac{1}{2}u_k(\xi) = \sum_{e=1}^{m} \left(\int_{\Gamma_e} p^*_{ik}(x,\xi)\,u_i(x)\,\mathrm{d}\Gamma_e - \int_{\Gamma_e} u^*_{ik}(x,\xi)\,p^0_i(x)\,\mathrm{d}\Gamma_e \right) +$$

$$+ \int_{\Omega} \varepsilon^*_{ijk}(x,\xi)\,\sigma^0_{ij}(x)\,\mathrm{d}\Omega \tag{3.5}$$

The procedure can be divided into the following steps:

1. Input the initial stress $\sigma^0_{ij} = 0$.
2. Solve the basic equations to obtain displacements on Γ_0.
3. Suppose $b_i = 0$ and $c_{kl} = \delta_{kl}$ the displacements on domain Ω are obtained and from Hooke's law we get σ^1_{ij}.
4. The real stress of quasilinear state follows (2.3).
5. Test the criterion of plasticity and define the distribution of σ^0_{ij} in such a manner that the sum $\sigma_{ij} + \sigma^0_{ij}$ fulfils the criterion.
6. If the correction is larger than the given error, go to step 2, otherwise terminate the process.

Jiang (1985) succeeded in finding the Galerkin tensor G_{ij}, which obeys the identity:

$$\int_P u^*_{ik} b_k \mathrm{d}P = \frac{b_j}{2\mu} \int_Q \left[2(1-\nu) \frac{\partial G_{ik}}{\partial x_j} - \frac{\partial G_{ij}}{\partial x_k} \right] n_j \mathrm{d}Q$$

where P is a domain, on which b_j is constant and nonvanishing, n_j is the j-th component of the unit outward normal to boundary Q of domain P.

By definition

$$\varepsilon^*_{ijk} = \frac{1}{2} \left(\frac{\partial u^*_{jk}}{\partial x_i} + \frac{\partial u^*_{ik}}{\partial x_j} \right)$$

The last two equations describe the effect of the initial stresses on a special domain in terms of the value of the integral along its boundary.

Algorithmization leads to triangulation similar to that of the FEM (in this case the internal elements – cells – do not increase the number of unknowns). The non-linearity criterion has to be satisfied on each internal element.

5. OPTIMIZATION ON HALFSPACE

Most structures are assessed *a posteriori*. The computational techniques together with modern numerical analysis allow substantially more prospective analytical processes. In this section we will deal with the optimization of admissible shapes of structures in connection with the contact problems.

The optimization problems can be formulated in various equivalent forms. For the purpose of applying the BEM a variational formulation with constraints appears to be relatively most suitable. The stationary point of the following functional is to be sought:

$$\Pi(\Omega) = \frac{1}{2} \int_\Omega \sigma_{ij} \varepsilon_{ij} \mathrm{d}\Omega - \int_{\Gamma_2} u_i g_i \mathrm{d}\Gamma + \lambda \left(\int_\Omega \mathrm{d}\Omega - V \right), \quad i, j = 1, 2 \tag{5.1}$$

g_i are components of the given tractions and, for the sake of simplicity, the volume weight forces are neglected.

The admissible domain Ω with a sufficiently smooth boundary Γ has volume V. The boundary consists of two parts, Γ_1 and Γ_2; in the first part the displacements are prescribed whereas in the second the tractions are given.

Using Gauss' theorem and static equations the functional (5.1) may be rewritten as follows:

$$\Pi(\Omega) = \frac{1}{2} \int_\Gamma u_i p_i \mathrm{d}\Gamma - \int_{\Gamma_2} u_i g_i \mathrm{d}\Gamma + \lambda \left(\int_\Omega \mathrm{d}\Omega - V \right), \quad i = 1, 2 \tag{5.2}$$

where p_i are components of the admissible tractions. Set

$$u = \{u_1, u_2, u_3\}^{\mathrm{T}}, \quad p = \{p_1, p_2, p_3\}^{\mathrm{T}}, \quad g = \{g_1, g_2, g_3\}^{\mathrm{T}},$$

then

$$\Pi(\Omega) = \frac{1}{2}\int_\Gamma u^{\mathrm{T}} p \,\mathrm{d}\Gamma - \int_{\Gamma_2} u^{\mathrm{T}}\, g \,\mathrm{d}\Gamma + \lambda\left(\int_\Omega \mathrm{d}\Omega - V\right), \tag{5.3}$$

Let us approximate vectors u and p as

$$u = \mathbf{D}\,\bar{u}, \quad p = \mathbf{D}\,\bar{p}$$

where $\mathbf{D}$ is the matrix of base functions and the barred quantities are free parameters of the problem. These parameters are related to the boundary only. The direct BEM leads to the relation (see Eq. (2.2)):

$$\mathbf{H}\bar{u} = \mathbf{G}\bar{p} \tag{5.4}$$

where both $\mathbf{H}$ and $\mathbf{G}$ are square matrices. Moreover, $\mathbf{G}$ is a regular matrix. This allows an alternative expression:

$$\bar{p} = \mathbf{G}^{-1}\,\mathbf{H}\,\bar{u} = \mathbf{Z}\,\bar{u}, \tag{5.5}$$

By inserting it into Eq. (5.3) one obtains

$$\Pi = \bar{u}^{\mathrm{T}}\mathbf{K}\bar{u} - \bar{u}^{\mathrm{T}}F + \lambda\left(\int_\Omega \mathrm{d}\Gamma - V\right) \tag{5.6}$$

where

$$\mathbf{K} = \frac{1}{2}\int_\Gamma \mathbf{D}^{\mathrm{T}}\mathbf{Z}\mathrm{d}\Gamma, \quad F = \int_\Gamma \mathbf{D}^{\mathrm{T}} g \mathrm{d}\Gamma$$

It remains to express volume V in terms of the parameters characterizing boundary Γ, which is a polygon (in 3D boundary Γ is generated by triangular parts of a plane). Choose an appropriate point C and connect it with each vertex of the polygon. We obtain N triangles T_k (in 3D tetrahedrons). The volume of domain Ω will then be

$$\int_\Omega \mathrm{d}\Omega = \sum_{k=1}^{N} \mathrm{meas}\,(T_k), \tag{5.7}$$

where meas (.) is an algebraic measure. The functional now has the form:

$$\Pi = \bar{u}^{\mathrm{T}}\mathbf{K}\bar{u} - \bar{u}^{\mathrm{T}}\mathbf{F} + \lambda\left(\sum_{k=1}^{N} \mathrm{meas}\,(T_k) - V\right) \tag{5.8}$$

Define the vector of internal parameters $p = \{p_1, \dots, p_M\}$, which can stand, for example, for the distances between C and the vertices. Variation with respect to u leads to the first system of Euler equations

$$\tfrac{1}{2}(\mathbf{K} + \mathbf{K}^{\mathrm{T}})\bar{u} = F, \quad \text{or} \quad \mathbf{K}\bar{u} = F \tag{5.9}$$

The derivative with respect to the shape parameters $p_l, l = 1, \ldots, M$, yields

$$\frac{1}{2}\bar{u}^{\mathrm{T}} \frac{\partial \mathbf{K}}{\partial p_l} \bar{u} = \lambda \sum_{k=1}^{N} \frac{\partial \operatorname{meas}(T_k)}{\partial p_l} = 0, \quad l = 1, \ldots, M$$

or

$$E_l + \lambda = 0. \tag{5.10}$$

The last condition for constant volume V yields

$$\sum_{k=1}^{N} \operatorname{meas}(T_k) = 0 \tag{5.11}$$

The system of Euler equations is strongly non-linear. But the following formulation seems to be auspicious and provides a relatively clear technical opinion.

E_l is related to the density of the surface deformation energy at node l. Eq. (5.10) requires the surface deformation energy at each node to be of the same value. For this reason the part of the boundary where E is greater than the real $-\lambda$ has to be extended, and in case of E is less than $-\lambda$ it has to be contracted. The current approximation of the real λ will be chosen in the following way:

$$-\lambda = \sum_{l=1}^{M} \frac{E_l}{M} \tag{5.12}$$

The new values of the shape parameters are given as follows:

$$p = \rho \frac{E_l}{\lambda} \tag{5.13}$$

where ρ is the given superrelaxation coefficient.

The numerical process can be divided into the following steps:

1. Define the initial configuration of the structure.
2. Boundary Γ is approximated by the boundary elements.
3. The stiffnes matrix $\mathbf{K}$ is generated for the current shape.
4. The current vector $\bar{u}$ is solved.
5. The value of E_l and the approximation of λ is computed for each p_l.
6. The shape of Γ is modified.
7. Test the convergence criterion, e.g.

$$\sum_{l=1}^{M} (p_l^0 - p_l^n)^2 < \varepsilon$$

where ε is the admissible error, p_l^0 are the old and p_l^n are the new values of the shape parameters. If the criterion is not satisfied, go to step 3, otherwise terminate the iteration.

We will show the application of the above procedures to the lining analysis using a substructure technique. Consider the configuration given in Fig. 1, where Γ describes the terrain, Γ_g is the part of the boundary where tractions are prescribed, on Γ_2 the displacements are given, and Γ_C is the contact line between the lining and the rock splitting domain Ω into two parts: into domain Ω_1 (surrounding rock) and Ω_2 (the lining).

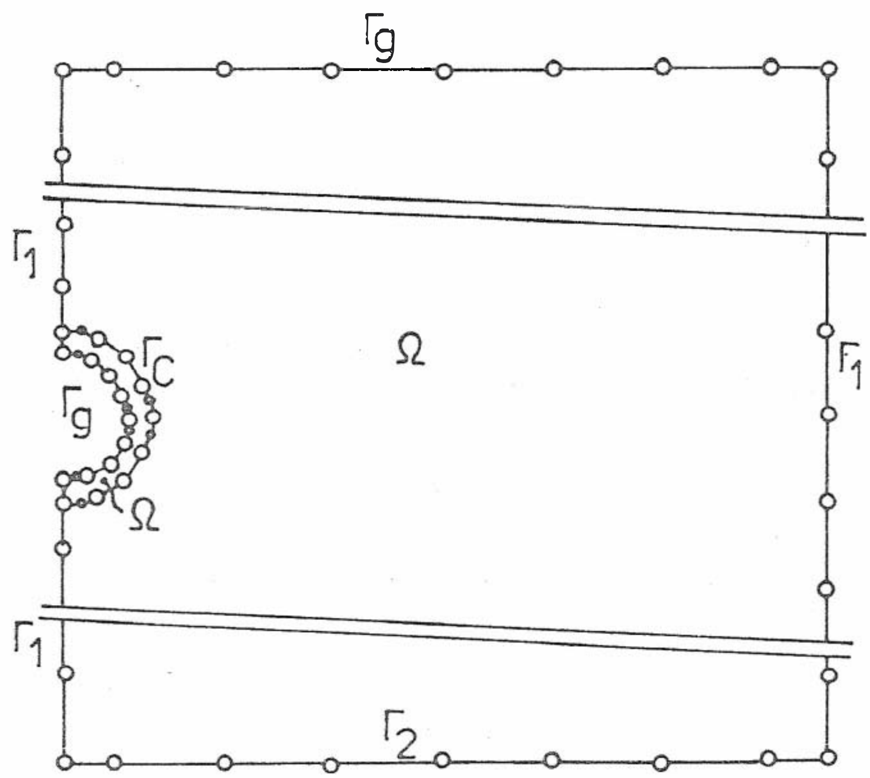

Fig. 1. Geometry of the problem and denotation.

After discretization of the relevant boundaries we obtain two systems of equations relating displacements $\boldsymbol{u}$ and tractions $\boldsymbol{p}$ with respect to the i-th domain (upper indices), $i = 1, 2$:

$$\begin{bmatrix} \mathbf{H}_{11}^1 & \mathbf{H}_{12}^1 \\ \mathbf{H}_{21}^1 & \mathbf{H}_{22}^1 \end{bmatrix} \begin{Bmatrix} u^1 \\ u_C^1 \end{Bmatrix} - \begin{bmatrix} \mathbf{G}_{11}^1 & \mathbf{G}_{12}^1 \\ \mathbf{G}_{21}^1 & \mathbf{G}_{22}^1 \end{bmatrix} \begin{Bmatrix} g^1 \\ p_C^1 \end{Bmatrix} = \begin{Bmatrix} F^1 \\ F_C^1 \end{Bmatrix} \tag{6.1}$$

$$\begin{bmatrix} \mathbf{H}_{11}^2 & \mathbf{H}_{12}^2 \\ \mathbf{H}_{21}^2 & \mathbf{H}_{22}^2 \end{bmatrix} \begin{Bmatrix} u_C^2 \\ u^2 \end{Bmatrix} - \begin{bmatrix} \mathbf{G}_{11}^2 & \mathbf{G}_{12}^2 \\ \mathbf{G}_{21}^2 & \mathbf{G}_{22}^2 \end{bmatrix} \begin{Bmatrix} p_C^2 \\ g^2 \end{Bmatrix} = \begin{Bmatrix} F_C^2 \\ F^2 \end{Bmatrix} \tag{6.2}$$

The lower index C denotes a variable defined at contact C, $\boldsymbol{g}$ denotes the prescribed tractions, $\boldsymbol{F}$ includes the volume weight effect. After some manipulation ($\boldsymbol{g}$ is a known vector) we arrive at

$$\begin{bmatrix} \mathbf{H}_{11}^1 & \mathbf{H}_{12}^1 \\ \mathbf{H}_{21}^1 & \mathbf{H}_{22}^1 \end{bmatrix} \begin{Bmatrix} u^1 \\ u_C^1 \end{Bmatrix} - \begin{bmatrix} \mathbf{G}_{12}^1 & p_C^1 \\ \mathbf{G}_{22}^1 & p_C^1 \end{bmatrix} = \begin{Bmatrix} F^1 & + & \mathbf{G}_{11}^1 & g^1 \\ F_C^1 & + & \mathbf{G}_{21}^1 & g^1 \end{Bmatrix} \tag{6.3}$$

$$\begin{bmatrix} \mathbf{H}_{11}^2 & \mathbf{H}_{12}^2 \\ \mathbf{H}_{21}^2 & \mathbf{H}_{22}^2 \end{bmatrix} \begin{Bmatrix} u_C^2 \\ u^2 \end{Bmatrix} - \begin{bmatrix} \mathbf{G}_{11}^2 & p_C^2 \\ \mathbf{G}_{21}^2 & p_C^2 \end{bmatrix} = \begin{Bmatrix} F_C^2 & + & \mathbf{G}_{12}^2 & g^2 \\ F^2 & + & \mathbf{G}_{22}^2 & g^2 \end{Bmatrix} \tag{6.4}$$

It is obvious that matrices $\mathbf{H}_{11}^1$ and $\mathbf{H}_{22}^2$ as well as matrices $\mathbf{G}_{ii}^k$, $i, k = 1, 2$, are regular. This is so because the elasticity problem can be solved. Canonical transformation (in practical cases we may use the Gaussian elimination – matrices $\mathbf{A}_{jj}^i$ need not be diagonal, which is due to the singularity of matrix $\mathbf{H}$):

$$\begin{bmatrix} \mathbf{A}_{11}^1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22}^1 \end{bmatrix} \begin{Bmatrix} u^1 \\ u_C^1 \end{Bmatrix} - \begin{bmatrix} \mathbf{G}_{12}^1 & p_C^1 \\ \mathbf{G}_{22}^1 & p_C^1 \end{bmatrix} = \begin{Bmatrix} B^1 \\ B_C^1 \end{Bmatrix} \tag{6.5}$$

$$\begin{bmatrix} \mathbf{A}_{11}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22}^2 \end{bmatrix} \begin{Bmatrix} u_C^2 \\ u^2 \end{Bmatrix} - \begin{bmatrix} \mathbf{G}_{11}^2 & p_C^2 \\ \mathbf{G}_{21}^2 & p_C^2 \end{bmatrix} = \begin{Bmatrix} B_C^2 \\ B^2 \end{Bmatrix} \tag{6.6}$$

where $\mathbf{0}$ is a null matrix. If equilibrium and compatibility hold along the contact line , i.e.

$$p_C^1 + p_C^2 = \mathbf{0} \qquad u_C^1 = u_C^2 \tag{6.7}$$

then:

$$([\mathbf{G}_{22}^1]^{-1}\, \mathbf{A}_{22}^1 + [\mathbf{G}_{11}^2]^{-1}\, \mathbf{A}_{11}^2)\; u_C^1 = [\mathbf{G}_{11}^2]^{-1}\; B_C^2 + [\mathbf{G}_{22}^1]^{-1}\; B_C^1 \tag{6.8}$$

and u_C follows immediately. In general, only the equilibrium along C holds, i.e.

$$\begin{aligned} \mathbf{A}_{22}^1\; u_C^1 &= \mathbf{G}_{22}^1\; p_C + B_C^1 \\ \mathbf{A}_{11}^2\; u_C^2 &= -\mathbf{G}_{11}^2\; p_C + B_C^2 \end{aligned} \tag{6.9}$$

where $p_C = p_C^1 = -p_C^2$.

Suppose now that the following conditions are prescribed along C:

$$\begin{gathered} [u]_n = u_n^1 - u_n^2 \geq 0, \quad p_n \geq 0, \quad [u]_n\, p_n = 0 \\ |p_t| \leq \mathcal{F} p_n + \mathcal{C} \\ |p_t| = \mathcal{F} p_n + \mathcal{C} \Rightarrow \exists \omega > 0, \quad [u]_t = -\omega p_t \end{gathered} \tag{6.10}$$

Then the Uzawa's algorithm can be used (see Céa, 1971):

1. Choose the initial value of p_C, say $p_C = \mathbf{0}$. Compute u_C^1 and u_C^2.

2. Verify the contact conditions at each node of the contact line. In case $[u]_n \leq 0$ put ($\varrho_n > 0$ is a given number):

$$p_t = \mathrm{P}_+[p_t - \rho_t[u]_t], \ \mathrm{P}_+[a] = a \ \text{ for } a \geq 0 \ \text{ else } \mathrm{P}_+[a] = 0$$

In case the second condition of (6.10) is valid the compatibility holds: $[u]_t = $ $= u_t^1 - u_t^2 = 0$. Choose a positive number ρ_t and put:

$$p_t = p_t - \rho_t[u]_t$$

If $|p_t| \geq \mathcal{F}p_n + \mathcal{C}$, then $[u]_t \neq 0$, $p_t = (\mathcal{F}p_n + \mathcal{C})$ sign p_t.

3. Vectors $\boldsymbol{u}_C^1$ and $\boldsymbol{u}_C^2$ are computed for this new generated vector $\boldsymbol{p}_C$. If the error criterion is violated, repeat step 2, otherwise compute the displacements on $\Gamma_p \cup \Gamma$.

4. Compute the displacements and stresses on Ω_1 and Ω_2. Note that because of the singular solution on Ω_1 it is necessary to ensure some regularization. We recommend the procedure with the artificial bolt after Janovský and Procházka (1980), see also Brož and Procházka (1985).

7. EXAMPLE

In this section we will test the distribution of tangential forces at the contact between the lining and surrounding rock according to Fig. 1. The boundary conditions are as follows: along Γ_1 zero tractions, along Γ_2 displacements in the x-direction and tractions in the y-direction are zero and on Γ_3 displacements are prescribed to be zero. The system is loaded by volume weight $\gamma = 0.024$ kN/m^3, $E_{\text{rock}} = 6000$ kN/m^2, $\nu_{\text{rock}} = 0.40$, $E_{\text{lin}} = 188{,}000$ kN/m, $\nu_{\text{lin}} = 0.15$. The physical law (6.10) is employed under the assumptions $\mathcal{F} = 0, \mathcal{C} = 1.4$ kN/m (thick line, p) and $c^0 = 1.0$ kN/m (bold line, p_t^0) (see Fig. 2).

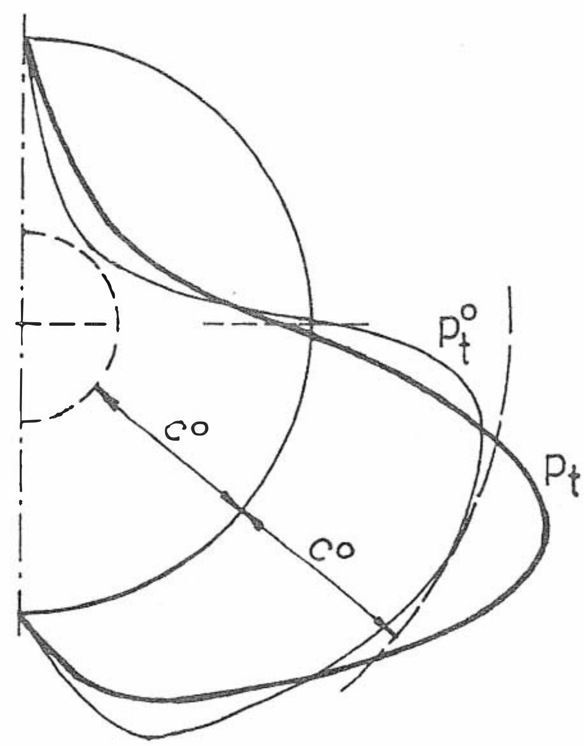

Fig. 2. Geometry and distribution of radial contact forces.

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OPTIMALIZACE A ŘEŠENÍ KONTAKTNÍCH PROBLÉMŮ V GEOTECHNICE METODOU OKRAJOVÝCH PRVKŮ

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V tomto příspěvku se zabýváme aplikací metody okrajových prvků na vybrané geotechnické problémy. Pozornost je věnována zejména napjatostní analýze podzemních konstrukcí s a bez ztužení (obezdívky apod.). Formulace vychází ze spojení kontaktních a optimalizačních problémů. Typický příklad demonstruje chování modelu.

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