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FREE GROUNDWATER LEVEL AT VERTICAL DAMS

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ABSTRACT. The free water level at vertical dams drops from the inlet to the outlet following a curve, whose analytical expression could probably not be found yet. In the accessible literature, this problem has been dealt with by (P.J.Polubarinova, 1952). Even there, the practically applicable equations for computing the free water level drop have not been quoted, expect a diagram for the determination of level height at one point, i.e. at the outlet from the dam. However, practically applicable equations for the calculation of free level drop can be obtained by the use of the velocity potential law, with the maximum error of 2%, as it results from the following study.

INTRODUCTION

The free drop (from the inlet H_S to the outlet H_0) in a vertical dam with width S is indicated in Figs.1 and 2. The lines of flow drop from the height Y_S to Y_0, upper free level from the height H_S to H_0. The discharge through the dam depends on the morphology of flow lines, i.e. on their shape and arrangement. The Fig.1 illustrates the two different internal arrangements of flow lines between points Y_S and Y_0: the weakly marked lines of flow have a gradually decreasing gradient, the last of them being almost parallel to the impermeable subsoil. Like in all natural processes, it can be assumed even here that the shape and the internal arrangement of flow lines will be optimum from the viewpoint of flow. Thus, such a shape of the free level, flow lines and their arrangement should be found, which would result in maximum discharge at the free level drop between H_S and H_0.

BASIC EQUATIONS

A certain constant water quantity flows through a flow tube, which is proportional to the difference of potential between the inlet and outlet. In Fig.2, it is the quantity δQ, flowing from the inlet Y_S to the outlet Y_0, the respective potential difference being U. As defined by the velocity potential, $dU = c.dL$. The velocity c has the direction of the tangent to the line of flow, being therefore related to the cross section f, perpendicular to the flow line. If the angle of tangent gradient is β, the length of the flow line element $dL = dX/\cos\beta$, and the flow cross section $f = \delta Y \cos\beta$. Velocity is $c = \delta Q/f$ and thus $dU = (\delta Q/\delta Y \cos^2\beta)\,dX$.

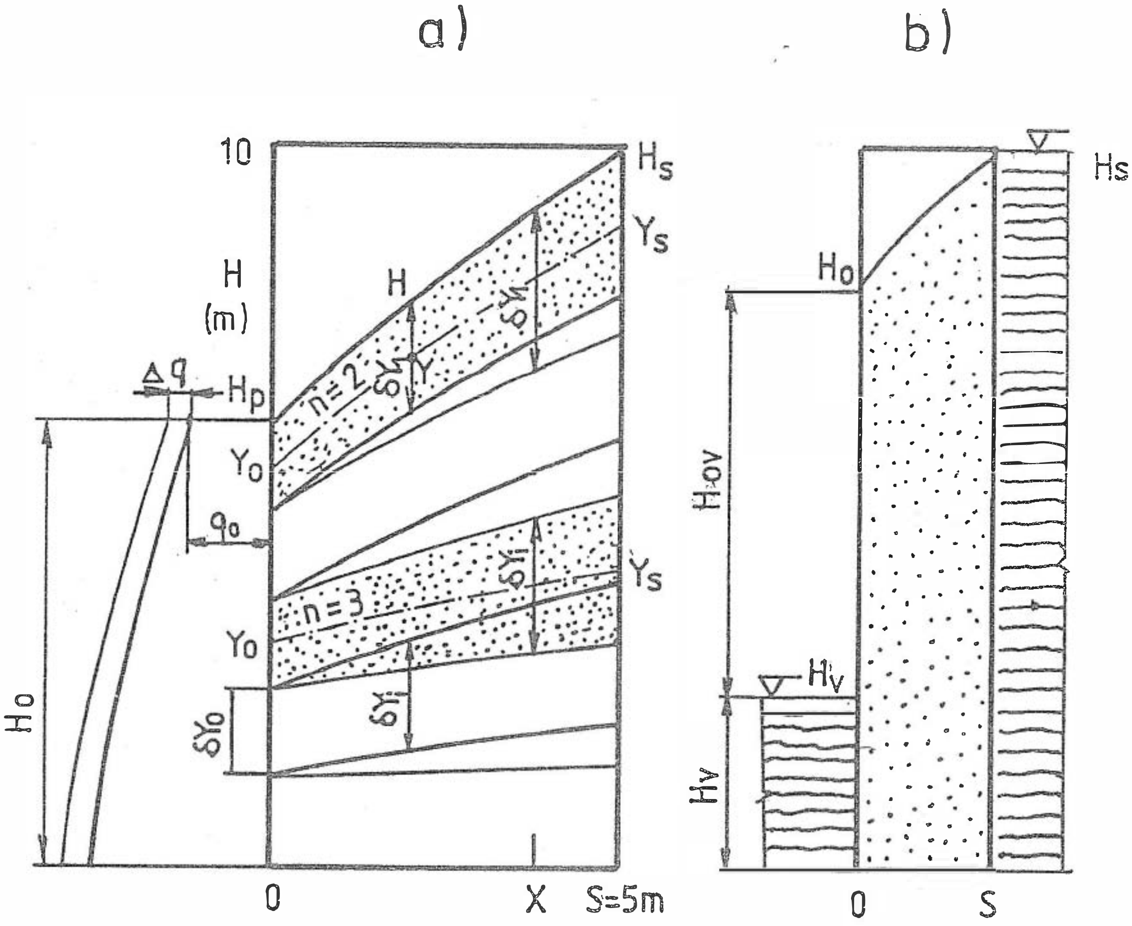

FIG.1. Morphology of flow lines

As $1/\cos^{\beta} = 1 + \tan^2\beta = \left[1 + (dY/dX)^2\right]$ then, if we denote $dY/dX = Y'$, the dependence between the potential drop, flow volume, and parameters of the given flow line can be expressed by the equation

$$U = \int_0^S \frac{\delta Q}{\delta Y}(1 + Y'^2)\, dX \tag{1}$$

INITIAL APPROXIMATION OF FLOW AT A VERTICAL DAM

The first assumption of this initial approximation is, that both the free level surface and the internal flow line are parabolas of the Dupit type, given by the equation

$$H^2 = H_0^2 + B_0 X \quad ; \quad B_0 = (H_S^2 - H_0^2)/S\,. \tag{2}$$

The constant B_0 results from the boundary conditions (for $X \doteq 0$, $H = H_0$, for $X = S$, $H = H_S$).

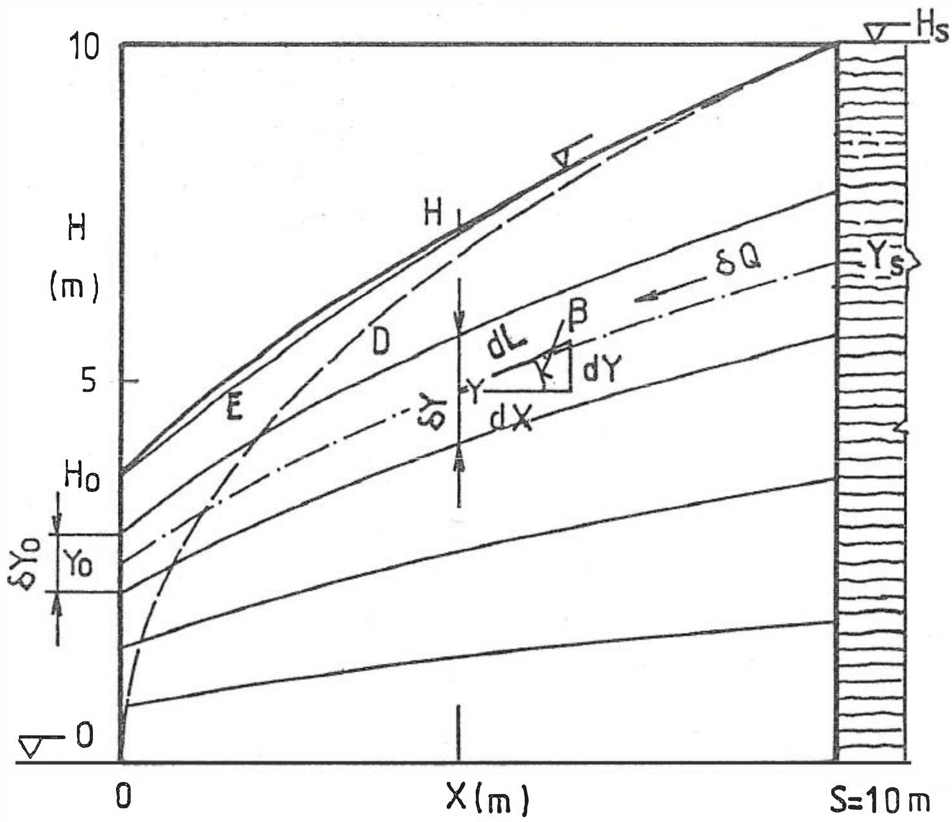

FIG.2. Scheme of the problem

The free level, illustrated in Fig.2 by the curve H, drops only to the height H_0. The level according to Dupit parabola (curve D in Fig.2) would drop as low as to the impervious subsoil.

The second assumption for this initial approximation is that each vertical section is passed through by the same number of flow tubes, which have, in the given section, the same width. For example, in Fig.1a this concerns five tubes, from which each has the width δY in the section H–X. Generally, the proportionalities $Y_0/H_0 = Y/H$, $\delta Y_0/H_0 = \delta Y/H$ hold true.

Each tube ($n = 2$), which issues, on the discharge side, from a point of the height of Y_0 will be defined, at such conditions, by the equation

$$Y^2 = Y_0^2 + BX \tag{3}$$

where

$$B = B_0(Y_0/H_0)^2 \quad \text{and} \quad dY/dX = Y' = B/2Y\,.$$

It results, from the second assumption, that $\delta Y = \delta Y_0(Y/Y_0)$. For the given tube, the values of δQ, δY_0 are constant; the equation (1) can thus be written in the form of

$$U = \frac{\delta Q}{\delta Y_0}\int_0^S Y_0\,\frac{1+Y'^2}{Y}\,dX \tag{4}$$

By integration and arrangement

$$\int_0^S Y_0 \frac{1+Y'^2}{Y} dX = H_0 R \quad ; \quad R = R_1 + R_2 (Y_0/H_0)^2 \tag{5}$$

$$R_1 = \frac{2S/H_S}{1+H_0/H_S} \quad ; \quad R_2 = (1 - H_0^2/H_S^2)(H_S/H_0 - 1)/(2S/H_S)$$

is obtained.

It is evident, from the physical interpretation of the equation (4), that the computed integral determines the value of the hydraulic resistance of the given tube, defined as the factor of proportionality between potential difference U and the flow $\delta Q/\delta Y_0$. By denoting the Darcy's filtration coefficient as k, the relevant potential difference will be $U = k(H_S - Y_0)$; the total discharge through the flow area will be computed by means of the equation (4):

$$Q = \int_0^{H_0} \frac{k(H_S - Y_0)}{R_1 + R_2(Y_0/H_0)^2} \frac{dY_0}{H_0} \tag{6}$$

The total flow through the dam is given by the Dupit formula:

$$Q_1 = kH_S^2/2S \tag{7}$$

It has been established that the flow volume depends only on the water levels at inlet and outlet, being completely independent on the flow height H_0 (Mls, 1988). The level height at the outlet is maintained, by pumping off, at the level of impervious subsoil, thus at zero – Fig.2. As it has been already mentioned, it is assumed that the spontaneously created flow field will be optimalized as far as the flow is concerned. The assumption of the starting approximation, i.e. the course of the free level and the internal arrangement of the flow lines, had therefore to be checked.

Course of the free level

By introducing $Y_0 = H_0$ into the equation (3), equation for the limiting flow line, which forms the free level, are obtained. The scope is to find a flow line, whose hydraulic resistance between points H_S and H_0 is minimal at the given conditions. The integral in the equation (4) resp. (5) represents the hydraulic resistance of the flow tube; its minimum values found as an extreme of the functional, from the condition:

$$\frac{\partial F}{\partial Y} - \frac{d}{dX}\left[\frac{\partial F}{\partial Y'}\right] = 0 \quad , \quad F = \frac{1+Y'^2}{Y}$$

The flow tube, which appertains to the free level and has a minimum resistance, is determined by the equation:

$$H = -\frac{1}{4C}(D + CX)^2 + \frac{1}{C} \tag{8}$$

where

$$D = -\frac{\sqrt{(S/2H_0)^2 + H_S/H_0 - 1}}{S/4H_0}$$

and

$$C = (1 - D^2/4)/H_0$$

as far as $dH/dX > 0, d^2H/dX^2 < 0$ within the interval of $0 \leq X \leq S$.

The hydraulic resistance of this flow line, computed from the equation (5), equals in such a case,

$$R_m = \int_0^S \frac{1 + Y'^2}{Y} dX = -2.\ln\left|\frac{CS + D - 2}{CS + D + 2} \cdot \frac{D + 2}{D - 2}\right| - CS \quad .$$

In Fig.2, the flow line with the minimum resistance is denoted by E, and does not differ much from the level H. For the given parameters $H_S = S = 10\,\mathrm{m}$, $H_0 = 3{,}8\,\mathrm{m}$, it can be computed, from the equation (5), that $R \doteq 2{,}135$; the flow rate at discharge $q = \delta Q/\delta Y_0$ results thus about 0,6% higher that the rate q_0, computed for the initial approximation. Generally, the increment $\triangle q = q_0(R/R_m - 1)$ – see Fig.1. The calculations proved that, for the internal flow lines, the differences in flow rates $\triangle q$ are only a little higher. Therefore, also the overall flow volume Q_m will be only a little higher than Q, computed for conditions of the initial flow approximation,

$$Q_m = Q + q_0 H_0(R/R_m - 1) \quad .$$

We are therefore justified to suppose that the spontaneously formed free level will be better described by the curve E, which is admittedly very close to the assumed course of H – see Fig.2 – but, with the same difference of $H_S - H_0$ exhibits lower hydraulic resistance and higher total flow volume Q_m. This problem – to find a flow line with minimum hydraulic resistance – is much similar to the problem of brachistochrone, known from mechanics.

Internal arrangement of flow lines

The Fig.1a illustrates two variants of the possible arrangement. The variant $n = 2$ represents the so-called regular arrangement – in each section $\delta Y_1 = \delta Y_2 = \cdots = cons.$ This does not hold true for the variant $n = 3$, where $\delta Y_1 \neq \delta Y_i$. In a regular arrangement, each flow line is defined by equation (3), and the following equation holds true for an arbitrary n:

$$Y^2 = Y_0^2 + B_0(Y_0/H_0)^n X \quad , \tag{9}$$

$$\delta Y = \delta Y_0 \left[\frac{n}{2} \cdot \frac{Y}{Y_0} + \frac{2-n}{2} \cdot \frac{Y_0}{Y}\right]$$

The second equation defines the dependence of the flow section on the exponent n. The flow lines of the variant $n = 3$ (Fig.1a) are computed from equation (9), for

$n = 3$, $H_S = 10\,\mathrm{m}$, $S = 5\,\mathrm{m}$, $H_0 = 6{,}2\,\mathrm{m}$. Parameters and computing results are often quoted in dimensionless quantities

$$s = S/H_S \quad , \quad h_0 = H_0/H_S \quad , \quad y_0 = Y_0/H_0 \quad , \quad y_S = Y_S/Y_0 \qquad (10)$$

For the total flow volume Q, the relation can be obtained from equations (6), (7), which can be arranged into the form of

$$\frac{Q}{Q_1} = \int_0^1 \frac{n(1-h_0^2)(1/h_0 - y_0)y_0^{n-2}}{2R}\,dy_0 \,. \qquad (11)$$

Q_1 is the real flow volume, given by the Dupit equation (7), R being the hydraulic resistance, which depends on the exponent n;

$$\text{for} \quad n > 2 \quad \text{is} \quad R = y_S - 1 + u(Z^2 - 1)\,[\arctan(y_S/u) - \arctan(1/u)]$$

$$\text{for} \quad n < 2 \quad \text{is} \quad R = y_S - 1 + 0{,}5u(Z^2 + 1)\ln\left|\frac{y_S - u}{y_S + u}\cdot\frac{1+u}{1-u}\right| \,,$$

where

$$Z = (1-h_0^2)y_0^{n-1}/(2sh_0u) \quad , \quad u = \sqrt{|2/n - 1|} \quad , \quad y_S = \sqrt{1 + (1/h_0^2 - 1)y_0^{n-2}}$$

and for $n = 2$, the following relation is obtained from the equation (11)

$$\frac{Q}{Q_1} = 2s(w\ \arctan w - 0{,}5h_0\ \ln(1 + w^2))/R_2 \qquad (12)$$

where

$$w = \sqrt{R_2/R_1} \quad , \quad R_1 = \frac{2s}{1 + h_0} \quad , \quad R_2 = (1-h_0^2)(1/h_0 - 1)/(2s)\,.$$

Values of R_1, R_2 are here the same as in the equation (5), being only expressed by means of dimensionless number (10). In this case ($n = 2$), the equation (11) has not to be integrated numerically. We can obtain, by its integration, the equation (12), which expresses the total flow volume, computed from conditions of initial approximation of flowing. Therefore, also the computed corrections are related to this flow.

Fig.3 illustrates the relationships of Q/Q_1 on the exponent n for three relative dam widths. These relationships were obtained by numerical integration of the equation (11), for parameters s, h_0, contained in the illustration. For example, a dam with relative width $s = 0{,}5$ will give Q_{max} with the exponent $n = 3$ (see Fig.3), i.e. the internal arrangement of flow lines according to the variant $n = 3$ in Fig. 1a. The total flow according to equation (12) depends also on the exponent n; it is evident, from Fig.3, that the maximum possible value Q_{max} is obtained, for given

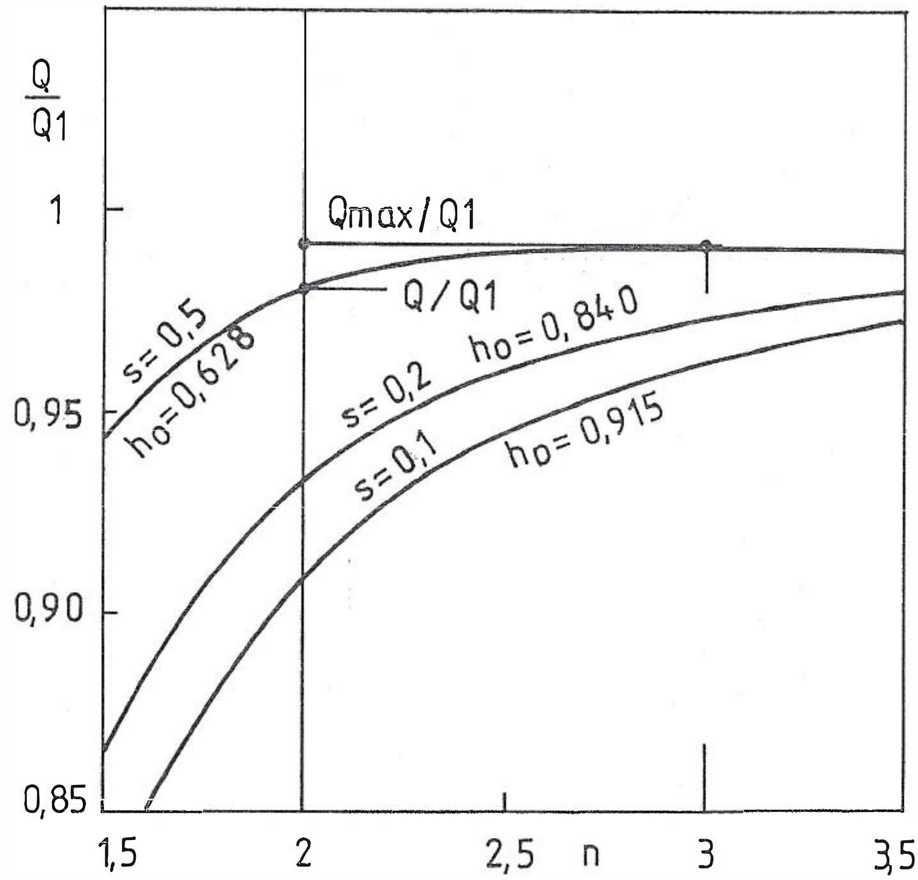

FIG.3. Diagram for the determination of flow correction

s, h_0 at $n = 3$. Considering the value of Q, we can obtain, in this case, from Fig.3, that for $s = 0{,}5$ and $h_0 = 0{,}628$

$$(Q_{max}/Q_1 - Q/Q_1)/(Q/Q_1) \doteq 0{,}0102\,.$$

It has been established, in the same way, for the interval $0{,}1 < s < 3{,}5$, that

$$\frac{Q_{max} - Q}{Q} \doteq 0{,}128\, e^{-5s} \tag{13}$$

holds true. This relationship will be used for the expression of the effect of the internal arrangement of flow lines on the flow through the dam.

COMPUTING OF THE OUTFLOW HEIGHT

The flow volume Q, computed by means of the law of velocity potential loss, is the same as the flow volume Q_1, computed by means of the Dupit relation (7), when the real flow field is known. It has been tried, in this study, to characterize this fact in the best possible way. This resulted in the relation for the determination of relative outflow height h_0, derived from the condition $Q = Q_1$, thus $Q/Q_1 = 1$,

$$2\,p\,s\,(w \arctan w - 0{,}5h_0 \ln(1+w^2))/R_2 = 1 \tag{14}$$

$$p = 1 + (R/R_m - 1)h_0 + 0{,}128e^{-5s}\,.$$

Information described in preceding chapters has been used for the derivation of the general correction factor p. The second member in the equation for p expresses the correction respecting the shape of the free level, the third one the correction for internal arrangement of flow lines, because both the level and the arrangement of flow lines differ in reality somewhat from assumptions of our approximation.

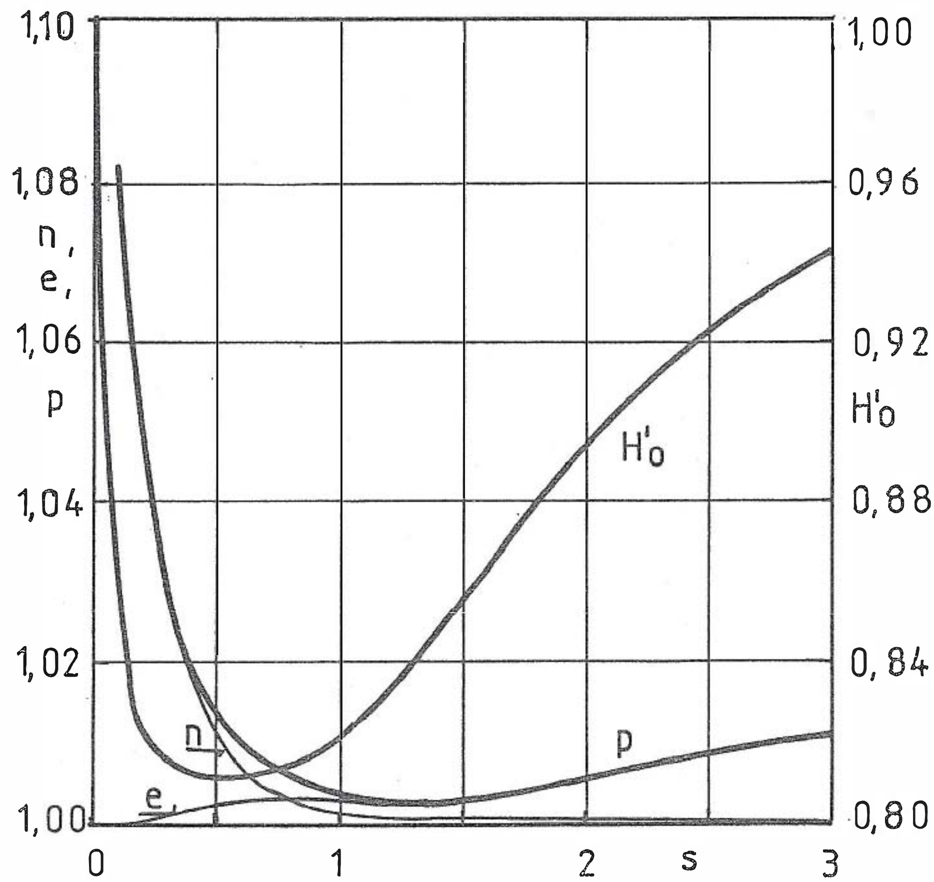

FIG.4: Dependence of the correction factor p and free level gradient H_0' on relative width s

Parameters in the equation (14) have already been defined in equations (5), (8), (10) and (12). The exists always a value h_0 for each relative dam width, which fulfills the equation (14). This is relative outflow height looked for, defined as the relation $h_0 = H_0/H_S$ – Fig.2. The Fig.4 illustrates the dependence of the correction factor p on the relative width s. The curve e characterizes the effect of the shape of the free level, the curve n the effect of internal arrangement of the flow lines. In slender dams, $s < 0{,}5$, the effect of the internal arrangement of the flow lines prevails, in broader dams, $s > 1$, the effect of the free level shape. The free level gradient at the outflow face, computed from equations (8), is $H_0' = -D/2$. Its dependence on relative width is illustrated by the curve H_0'. It is interesting to see that even if the computation of gradient issues only from the requirement of the optimum shape, of the free level, it fulfills also the condition, derivate from differential equations of planar flow, that the free level gradient H_0' must always be $H_0' < 1$ (Hálek and Švec, 1973).

The dependence of the relative outflow height h_0 on the relative width is tabulated numerically in Tab.1 and graphically in Fig.5., where weak lines illustrate the mentioned relationships also for cases,when the lower lever $H_V \neq 0$ – Fig.1b. Similarly as the relative outflow height $h_0 = H_0/H_S$, other relative heights $h_{0V} = H_{0V}/H_S$, $h_V = H_V/H_S$ can here be defined.

The approximate relation (15) holds true for all curves in Fig.5:

$$h_{0V} \doteq (1 - h_V)\, e^{-f(s)} \quad ; \quad f(s) = s\, \frac{0{,}96 - 0{,}12\,|s - 0{,}8|}{(1 - h_V)} \tag{15}$$

The relationship for $h_V = 0$ has been obtained by regression from computed values (Tab.1), for the others, up to $h_V = 0{,}8$, also by means of the analysis of hydraulic conditions in a very slender dam, for $s \to 0$.

TAB.1. Relative outflow height h_0
of vertical dam with relative width s

s	h_0	s	h_0	s	h_0
0,1	0,915	1,1	0,358	2,1	0,189
0,2	0,840	1,2	0,331	2,2	0,180
0,3	0,767	1,3	0,308	2,3	0,171
0,4	0,696	1,4	0,287	2,4	0,163
0,5	0,628	1,5	0,268	2,5	0,155
0,6	0,566	1,6	0,252	2,6	0,148
0,7	0,512	1,7	0,247	2,7	0,142
0,8	0,464	1,8	0,223	2,8	0,136
0,9	0,424	1,9	0,211	2,9	0,131
1,0	0,388	2,0	0,199	3,0	0,125

EXAMPLE

The dam has the width $S = 10\,\mathrm{m}$, the higher level is $H_S = 50\,\mathrm{m}$ above impervious subsoil; the water is pumped off at the outflow, $H_V = 0$. Let the outflow height and the free water surface level H in the dam be determined at the distance of $X = 2\,\mathrm{m}$ from the outflow wall. The relative width $s = 10/50 = 0{,}2$. As result from equations (14), (15), or from diagram in Fig.5, the values of $h_0 = 0{,}84$ and thus $H_0 = 42\,\mathrm{m}$. At the distance of $X = 2\,\mathrm{m}$ from the inlet, the free level height

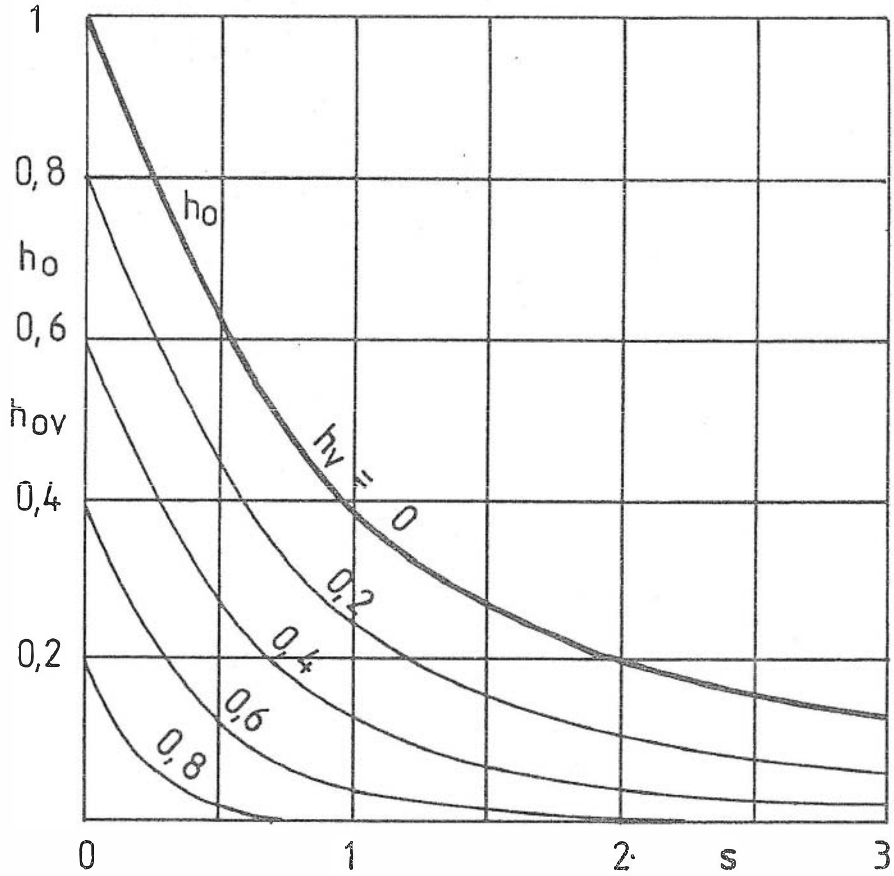

FIG.5. Diagram for the determination of the outflow height

is $H = 43{,}63\,\mathrm{m}$ according to equation (8), and $H = 43{,}71\,\mathrm{m}$ according to equation (2). If the lower surface level, e.g. $H_V = 0\,\mathrm{m}$ above impervious subsoil, then the equation (15) or the diagram in Fig.5 (for $h_V = 20/50 = 0{,}4$) would be used to determine $h_{0V} \doteq 0{,}45$, $H_{0V} \doteq 22{,}5\,\mathrm{m}$. In equations (2), (8), the symbolH_0 has the same significance, indicating the free water surface level above impervious subsoil; here it is $H_0 = 20 + 22{,}5 = 42{,}5\,\mathrm{m}$.

It can be computed, from both equations, for h_0, that the free level height at the distance of $X = 2\,\mathrm{m}$ from the outflow wall is $H \doteq 44{,}1\,\mathrm{m}$ above impervious subsoil.

CONCLUSIONS

It has been shown that the approximation of the course on a free water level in a vertical dam by means of Dupite-type curves gives very exact results. The problem of the free level is discussed in some detail in the work of (Polubarinova, 1952). The theoretical solution of this problem is so complex that it can be only roughly sketched on 10 text pages. The procedure of numerical calculations, from which the graph for the determination of the outflow height was constructed in (Polubarinova, 1952), could therefore not yet be derived. No other than a graphical comparation with result of the described solution was therefore available. However, any evident deviations could not be established. For graphs, the error of 1–2% is admitted and therefore quoted this value at the beginning of this study.

It is therefore possible,by means of the law of velocity potential loss, to determine not only the height, but also the course of the free level. Nevertheless, this is not just an approximate calculations usually understood in technical practice. Results of the here suggested computation would be entirely precise for the real shape and arrangement of flow lines.

LIST OF THE USED SYMBOLS

H_S (m) .... height of the steady level above impervious layer
H_V (m) .... height of the lowered level above impervious layer
S (m)...... dam width
Q_1 ($\mathrm{m\,s^{-1}}$) . flow rate through the dam
H_{0V} (m) ... difference between H_0 and H_V
H_0 (m) .... height of the free level at outflow
U ($\mathrm{m^2\,s^{-1}}$) . difference of potentials at the entrance and outlet of the given flow

REFERENCES

Bush K.F. and Luckner L. (1972), *Geohydraulik*, Leipzig. (in German)
Hálek V. and Švec J. (1973), *Groundwater Hydraulics*, Academia, Prague. (in Czech)
Mls H. (1988), *Groundwater Hydraulics*, ČVUT, Prague. (in Czech)
Mucha I. and Šestakov V.M. (1987), *Hydraulics of groundwaters*, ALFA, Bratislava. (in Czech)
Polubarinova P.J. (1952), *Theory of Flow of Groundwater*, GITTL, Moscow. (in Russian)
Rektorys K. (1968), *Review of Applied Mathematics*, SNTL, Prague. (in Czech)

VOLNÁ HLADINA PODZEMNÍ VODY U SVISLÝCH HRÁZÍ

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Volná hladina ve svislé hrázi klesá od výtoku po nějaké křivce, jejíž analytické řešení dosud pravděpodobně nebylo nalezeno. Z dostupné literatury je tento problém podrobněji popsán v knize (Polubarinová, 1952). Ani zde však nejsou odvozeny rovnice pro pokles volné hladiny, jen diagram pro určení výronové výšky, tj. výšky volné hladiny na výtoku z hráze. Analytickou závislost pro pokles volné hladiny je však možné získat pomocí zákona o rychlostním potenciálu, a to s maximální chybou 2%, jak vyplyne z výsledků následující studie. Tímto způsobem je možné určovat průběh volné hladiny i v okolí kruhových studní, kde analytické řešení neexistuje a sotva kdy bude nalezeno. Znalost výšky výronové plochy a průběhu volné hladiny má značný praktický význam tehdy, když jde o studny velkých rozměrů, jako jsou jámy důlních děl, využívaných jako netradiční zdroje užitkové, případně i pitné vody.