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ON ELASTIC–PLASTIC THEORY OF GEOMATERIALS

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ABSTRACT. The elastic–plastic theory is discussed in this paper provided the plastic deformations fulfil the condition that the plastic deformation may be introduced. It is, for example, in case the given material can be deformed from any state into the state with vanishing stress by an elastic way. The theory of the infinitesimal elastic and elastic–plastic deformations are derived from the general theory.

KEY WORDS: continuum mechanic, elastic–plastic theory

1. INTRODUCTION

At present the behaviour of geomaterials is described especially by the elastic–plastic constitutive relations. The elastic–plastic theory for the infinitesimal deformations was established in 19^{th} century, but the theory of the finite elastic–plastic deformations is not so far satisfactory, to the author's knowledge. It is assumed in the theory of the infinitesimal deformations that the total strain tensor is the sum of the elastic and the plastic parts and the rate of plastic strain tensor is given by the stress tensor. The attempt for the description of the elastic–plastic material for the finite deformations by a such decomposition of the total strain tensor is in the paper [Green and Naghdi 1965]. But it is not quite evident from the experimental standpoint how to carry out this decomposition.

A completely different description of the elastic–plastic continuum is used in the paper [Del Piero 1975]. It is not necessary to introduce the plastic deformations in this description but in fact, any vector field in the space total strain–stress is given, which determines the time evolution of the material element. This description of the elastic–plastic material follows the description of the material element according to [Noll 1972] in which the mechanical process of a compatible pair configuration–process is considered as a constitutive relation. This theory is very remarkable from mathematical point of view but it says practically nothing about the constitutive relations, i.e. about the above mentioned vector fields.

It looks like that the combination of these two theories, i.e. including the plastic deformations into the elastic–plastic theory according to [Del Piero 1975] could contribute for the next development of this theory at least in the case, when there are plastic displacements to the suitably defined plastic deformations. It means that it is always possible to any plastic strain tensor to find the displacement in such a way that the plastic strain tensor is related to the plastic displacement from an initial state. If we define the elastic deformation as that of corresponding to the displacement from the plastic state, we obtain, in fact, the usual elastic theory only with a difference that the reference system is changed. It is a certain interstage between the Lagrange's and the Euler's descriptions of the continuum.

But, in order to introduce the plastic coordinates, the equations describing the evolution of plastic strain have to fulfil special relations.

We restrict our considerations to the geomaterials only, but this theory may be generalized to any deformative processes where there is an external force field, which under condition of no change of the plastic deformations causes a vanishing of the stress in the whole body. The condition which we suppose for the time evolution of the plastic strain tensor is only the sufficient condition for this property. Therefore, we will speak rather about the materials which fulfil this condition at any point and any time.

2. BASIC CONCEPTS

A body $\mathcal{B}$ from the continuum mechanics point of view is a compact C^∞–manifold in $\mathbb{R}^3$. We will describe it in the coordinates[1] ξ^α, $\alpha = 1, 2, 3$. The motion of the body is described in this case by functions χ^i in such a way that the position of the point ξ^α at the time t, i.e. x^i, is given as

$$x^i = \chi^i(\xi, t)\,. \tag{2.1}$$

We define the deformation gradient of the motion as follows

$$F^i_\alpha = \frac{\partial \chi^i}{\partial \xi^\alpha}\,. \tag{2.2}$$

and the velocity and the acceleration by

$$V^i(\xi, t) = \frac{\partial \chi^i}{\partial t}(\xi, t) \tag{2.3}$$

and

$$A^i(\xi, t) = \frac{\partial V^i}{\partial t}(\xi, t) = \dot{V}^i(\xi, t)\,, \tag{2.4}$$

respectively.

Further, the time independent mass density $\varrho_0(\xi)$, the vector field of the external forces b^i and the stress tensor $S^{i\alpha}$ are given. The motion equation has the form

$$\varrho_0 \dot{V}^i = \varrho_0 b^i + \frac{\partial S^{i\alpha}}{\partial \xi^\alpha} \tag{2.5}$$

and

[1] We will solely use the Cartesian coordinates in the whole paper

$$F^k_\alpha S^{i\alpha} = F^i_\alpha S^{k\alpha} . \tag{2.6}$$

The constitutive relations for the $S^{i\alpha}$ have to be added to these equations, i.e. we have to determine a dependance of the stress tensor $S^{i\alpha}$ on the motion χ^i. These equations together with the initial conditions determine the motion of the body in the Lagrange's coordinates.

It is well known the form of the equations (2.5) and (2.6) in the Euler's coordinates which are connected to the reference system of the instantaneous state of the body, i.e. in the coordinates x^i. Therefore we do not cast them here.

More details about the above introduced concepts and the reasons for their introduction may be find in any book about the rational continuum mechanics, see e.g. [Truesdell 1984] or [Leigh 1968].

3. Definition of Plastic Deformation

In the theory of the elastic–plastic continuum is assumed that there is an elastic range in which the plastic deformations are unchanging. In contrast with the usual elastic–plastic theory we will not decompose the total strain tensor into the elastic and the plastic parts, in which the total strain tensor is defined as the square of the displacement gradient. We rather decompose the total motion gradient $\boldsymbol{F}$. Thus, we write

$$F^i_\alpha(\xi, t) = X^i_k P^k_\alpha(\xi, t) , \tag{3.1}$$

where $\boldsymbol{P}$ corresponds to the plastic and $\boldsymbol{X}$ to the elastic parts of the total deformation. This decomposition is chosen for the reason that we obtain the equation (3.1) as the derivative of the relation $x^i = \psi^i(\pi(\xi, t), t)$.

We suppose that the stress tensor $S^{i\alpha}$ is a function of the variables $\boldsymbol{F}$ and $\boldsymbol{P}$. Thus

$$S^{i\alpha} = S^{i\alpha}(\boldsymbol{F}, \boldsymbol{P}) . \tag{3.2}$$

Further we suppose that there is the yield surface

$$\varphi(\boldsymbol{F}, \boldsymbol{P}) = 0 \tag{3.3}$$

such that the rate of the plastic deformations vanishes for $\boldsymbol{F}$ and $\boldsymbol{P}$ which fulfil the condition $\varphi(\boldsymbol{F}, \boldsymbol{P}) < 0$. A change of the $\boldsymbol{P}$ can occur only if

$$\varphi(\boldsymbol{F}, \boldsymbol{P}) = 0 \qquad \text{and} \qquad (V\varphi) = \frac{\partial \varphi}{\partial F^i_\alpha} \frac{\partial V^i}{\partial \xi^\alpha} > 0 . \tag{3.4}$$

Using the method from [Del Piero 1975] under the assumption that the processes are the changes of the total deformations and the compatible pairs are $(\boldsymbol{F}, \boldsymbol{P})$, i.e. the total deformations–plastic deformations, we obtain

$$\dot{P}^i_\alpha(\xi, t) = (V\varphi)\, M^i_\alpha(\xi, t) , \tag{3.5}$$

under the condition (3.4) and

$$\dot{P}^i_\alpha(\xi, t) = 0 \tag{3.6}$$

for

$$\varphi(\boldsymbol{F}, \boldsymbol{P}) < 0 \qquad \text{or} \qquad (V\varphi) \leq 0\,.$$

On the yield surface $\varphi(\boldsymbol{F}, \boldsymbol{P}) = 0$, at the same time, M^i_α fulfils the condition

$$\frac{\partial \varphi}{\partial P^i_\alpha} M^i_\alpha + 1 = 0\,, \tag{3.7}$$

what follows from the time derivation of the function φ.

What we require it is

$$P^i_{\alpha,\beta}(\xi, t_0) = P^i_{\beta,\alpha}(\xi, t_0) \tag{3.8}$$

for the initial plastic deformations and

$$\dot{P}^i_{\alpha,\beta} = \dot{P}^i_{\beta,\alpha}\,, \tag{3.9}$$

where $\dot{P}^i_{\alpha,\beta}$ denotes the derivative of the P^i_α with respect to ξ^β. It is well known that these conditions guarantee the existence of a function $\pi^i(\xi, t)$ such that

$$P^i_\alpha = \frac{\partial \pi^i}{\partial \xi^\alpha}$$

We call

$$p^i = \pi^i(\xi, t) \tag{3.10}$$

the plastic coordinates.

In fact, we have three systems of coordinates:

(1) the coordinates ξ^i in the Lagrange's reference system;
(2) the coordinates x^i in the Euler's reference system and
(3) the coordinates p^i in the plastic reference system.

The transformation relations are

$$x^i = \chi^i(\xi, t) \qquad \text{or} \qquad x^i = \psi^i(p, t) \tag{3.11}$$

and

$$p^i = \pi^i(\xi, t)\,, \tag{3.12}$$

where

$$\psi = \chi \circ \pi^{-1}\,. \tag{3.13}$$

From the equation (3.5) and (3.9) it is evident that

$$\left[(V\varphi)\, M^i_\alpha\right]_{,\beta} = \left[(V\varphi)\, M^i_\beta\right]_{,\alpha} \tag{3.14}$$

has to hold for any α and β. We may understand this equation either as a restriction to the possible rates of the total and plastic deformations for given M_α^i or, how we will proceed, as a further restriction to the constitutive relations for M_α^i if we assume that all the deformations agreeing with the relations (3.3) an (3.7) are possible. Some details about these equations can be found in Appendix.

It is quite natural to describe the motion of the body by means of the plastic coordinates p^i, since displacements from these coordinates are, in fact, the elastic displacements and the proposition that the stress for fixed plastic deformations (in elastic range) is a function of the gradient of the elastic displacements, has a good sense. In fact, it was the starting sense of the whole construction. In the rest of this paper we suppose that there exist such plastic coordinates and we transform the motion equations into these coordinates.

4. Motion Equations in the Plastic Coordinates

The motion equations in the Lagrange's coordinates are given by the relations (2.5) and (2.6) together with the fact that ϱ_0 is time independent. In this section we transform these equations to the plastic coordinates given by relation (3.10).

First, we introduce the velocity of the plastic displacement by

$$U^i(\xi,t) = \frac{\partial \pi^i}{\partial t}(\xi,t) \qquad \text{or} \qquad u^i(p,t) = U^i \circ \pi^{-1}(p,t) \tag{4.1}$$

and for simplicity we denote $\boldsymbol{Y}$ the inverse matrix to $\boldsymbol{P}$ and we consider it as the function of p and t. Thus

$$Y_r^\rho = \frac{\partial(\pi^{-1})^\rho}{\partial p^r}(p,t)\,. \tag{4.2}$$

The material derivative of any function $f(p,t)$, i.e. f', is

$$f'(p,t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p^k}\,u^k \tag{4.3}$$

Now we define the mass density ϱ_p by

$$\varrho_p(p,t) = \left(\varrho_0 P^{-1}\right) \circ \pi^{-1}(p,t)\,, \tag{4.4}$$

where

$$P = \det(\boldsymbol{P})\,. \tag{4.5}$$

It is easy to show that the mass density fulfils the continuity equation

$$\varrho_p' + \varrho_p\,\frac{\partial u^k}{\partial p^k} = 0\,. \tag{4.6}$$

If we introduce the stress tensor by

$$S_p^{ir}(p,t) = (P^{-1}P_\alpha^r S^{i\alpha}) \circ \pi^{-1}(p,t)\,, \tag{4.7}$$

the motion equations (2.5) and (2.6) have the form

$$\varrho_p v^{i\prime} = \varrho_p b^i + \frac{\partial S_p^{ir}}{\partial p^r} \tag{4.8}$$

and

$$\Sigma_p^{rs} = X_i^r S_p^{is} = \Sigma_p^{sr} \tag{4.9}$$

As we do not deal with the thermodynamics here, we do not transform the equations for the internal energy and others thermodynamic quantities to the plastic coordinates. But we should recommend to the reader also to carry out these transformations.

In the next we use in the constitutive relations instead of the variables $\boldsymbol{F}$ and $\boldsymbol{P}$ the plastic variables $\boldsymbol{X}$ and $\boldsymbol{Y}$, which we consider as functions of the plastic coordinates p and time t. Thus we assume that

$$\Sigma_p^{rs} = \Sigma_p^{rs}(\boldsymbol{X}, \boldsymbol{Y}) \tag{4.10}$$

and the yield surface is

$$\varphi(\boldsymbol{X}, \boldsymbol{Y}) = 0 \,. \tag{4.11}$$

It is easy to show that

$$(V\varphi) = \frac{\partial \varphi}{\partial X_k^i} \frac{\partial v^i}{\partial p^k} \,. \tag{4.12}$$

holds in the plastic coordinates. The equation (3.5) for the rate of the gradient of the plastic displacement has the form

$$\frac{\partial u^i}{\partial p^k} = (V\varphi) m_k^i \,, \tag{4.13}$$

where

$$m_k^i = M_\alpha^i Y_k^\alpha(p,t) \tag{4.14}$$

and finally, the condition (3.7) for M_α^i is

$$\left[Y_r^\alpha \frac{\partial \varphi}{\partial Y_s^\alpha} + X_r^k \frac{\partial \varphi}{\partial X_s^k} \right] m_s^r = 1 \,. \tag{4.15}$$

For completeness, we cast here the derivatives of the functions ξ^α and x^i, if we consider them a functions of the plastic coordinates p and time t. The following relations hold

$$\frac{\partial \xi^\alpha}{\partial p^k}(p,t) = Y_k^\alpha(p,t) \tag{4.16}$$

$$\frac{\partial \xi^\alpha}{\partial t} + Y_k^\alpha u^k = 0 \tag{4.17}$$

$$\frac{\partial x^i}{\partial p^k} = X_k^i \tag{4.18}$$

$$\frac{\partial x^i}{\partial t} = v^i - X_k^i u^k \,, \tag{4.19}$$

where (4.16) is the definition of $\boldsymbol{Y}$, (4.17) says, that ξ^α are the Lagrange's coordinates, (4.18) is the definition of $\boldsymbol{X}$ and (4.19) transforms the velocity into the plastic coordinates.

It is evident from the equation (4.13) that

$$\left[(V\varphi)\, m_k^i\right]_{,j} = \left[(V\varphi)\, m_j^i\right]_{,k} \tag{4.20}$$

holds for any j and k. It is necessary to understand this equation as the equation (3.14) from the previous section.

5. Principle of Material Indifference and others Symmetries

The principle of the material indifference [Leigh 1968] says that the constitutive relations are invariant with respect to arbitrary uniform rigid body motion, i.e. with respect to transformations

$$\tilde{x}^i = Q_k^i(t)\, x^k + c^i(t)\,, \tag{5.1}$$

where $\boldsymbol{Q}(t)$ is orthogonal, i.e. t

$$\boldsymbol{Q}(t)\boldsymbol{Q}^{\mathrm{T}}(t) = \boldsymbol{Q}^{\mathrm{T}}(t)\boldsymbol{Q}(t) = \boldsymbol{I}$$

holds.

It follows from this principle that the functions
respect to the transformation (5.1). Since we have assumed that these functions depend only on $\boldsymbol{X}$ and $\boldsymbol{Y}$, the equations

$$\Sigma_p^{ik}(\boldsymbol{X},\boldsymbol{Y}) = \Sigma_p^{ik}(\boldsymbol{QX},\boldsymbol{Y}) \tag{5.2}$$

$$\varphi(\boldsymbol{X},\boldsymbol{Y}) = \varphi(\boldsymbol{QX},\boldsymbol{Y}) \tag{5.3}$$

$$m_k^i(\boldsymbol{X},\boldsymbol{Y}) = m_k^i(\boldsymbol{QX},\boldsymbol{Y}) \tag{5.4}$$

hold, according to this principle. It follows from the paper [Smith 1971] that these functions are dependent on the quantities $\boldsymbol{P}$ and the elastic strain tensor

$$E_{rs} = X_r^i X_s^i\,. \tag{5.5}$$

only.

Under these assumptions we obtain

$$(V\varphi) = \frac{\partial\varphi}{\partial E_{rs}}\,\hat{E}_{rs}\,, \tag{5.6}$$

where

$$\hat{E}_{rs} = E'_{rs} + E_{rk}\,\frac{\partial u^k}{\partial p^s} + E_{sk}\,\frac{\partial u^k}{\partial p^r} \tag{5.7}$$

The condition (4.15) gets the form

$$\left(Y_r^\alpha \frac{\partial \varphi}{\partial Y_s^\alpha} + 2\, E_{ri} \frac{\partial \varphi}{\partial E_{si}}\right) m_s^r = 1\,. \tag{5.8}$$

Under the assumption that $Y_r^\alpha \dfrac{\partial \varphi}{\partial Y_s^\alpha} m_s^r \neq 0$ holds, we obtain

$$u_k^i = -\left[Y_r^\alpha \frac{\partial \varphi}{\partial Y_s^\alpha} m_s^r\right]^{-1} \frac{\partial \varphi}{\partial E_{rs}} E'_{rs} m_k^i\,. \tag{5.9}$$

The constitutive relations become more simple, if we assume that the material has additional symmetries. The most important in these symmetries is the isotropy, it means that it is possible to find the Lagrange's coordinates in such a manner that the properties of the materials are unchanged with respect to their orthogonal transformations, i.e. with respect to transformations

$$\tilde{\xi}^\alpha = Q_\beta^\alpha \xi^\beta\,, \tag{5.10}$$

where $\boldsymbol{Q}$ is orthogonal. In this case

$$\tilde{Y}_i^\alpha = \frac{\partial \tilde{\xi}^\alpha}{\partial p^i} = Q_\beta^\alpha Y_i^\alpha = (\boldsymbol{QY})_i^\alpha \tag{5.11}$$

holds and thus

$$\varphi(\boldsymbol{E}, \boldsymbol{Y}) = \varphi(\boldsymbol{E}, \boldsymbol{QY}) \tag{5.12}$$

$$\Sigma_p^{rs}(\boldsymbol{E}, \boldsymbol{Y}) = \Sigma_p^{rs}(\boldsymbol{E}, \boldsymbol{QY}) \tag{5.13}$$

$$m_k^i(\boldsymbol{E}, \boldsymbol{Y}) = m_k^i(\boldsymbol{E}, \boldsymbol{QY})\,. \tag{5.14}$$

It follows from these relations [Smith 1971] that these functions are dependent only on $\boldsymbol{E}$ and $\boldsymbol{B}$, where

$$B_{rs} = Y_r^\alpha Y_s^\alpha \tag{5.15}$$

is the plastic strain tensor.

We obtain from the equation (4.15) that $\boldsymbol{m}$ fulfils the condition

$$2\left[\frac{\partial \varphi}{\partial B_{st}} B_{rt} + \frac{\partial \varphi}{\partial E_{st}} E_{rt}\right] m_s^r = 1 \tag{5.16}$$

and it follows from (4.13) that the equation for the rate of the plastic strain tensor has the form

$$B'_{rs} = -(V\varphi)\left[B_{rt} m_s^t + B_{st} m_r^t\right] \tag{5.17}$$

Note that the strain tensors $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy the additional conditions, the compatibility equations, which guarantee that it is possible to write them in the form (5.5), and (5.15). The restricting relations to $\boldsymbol{m}$ follows from these compatibility equations and the equation (5.17).

Further symmetry, which can be meaningful for the elastic–plastic material, is the symmetry respecting all unitary transformations of the Lagrange's coordinates. In fact, this symmetry says that the plastic deformations behave like the deformations of a fluid and thus φ, Σ_p^{ik} and m_k^i depend only on the elastic strain tensor $\boldsymbol{E}$ and $P = \det \boldsymbol{P}$. In such a case the equation (4.15) is

$$Y\frac{\partial\varphi}{\partial Y}m_r^r + 2\,E_{rt}\frac{\partial\varphi}{\partial E_{st}}m_s^r = 1 \tag{5.18}$$

and the equation for the rate of the plastic volume deformation has the form

$$Y' = -Y(V\varphi)m_r^r\,. \tag{5.19}$$

It is evident that it is possible to obtain the additional restrictions to the constitutive relations, if we assume further symmetries with respect to any group of transformations of the plastic coordinates. But, since we do not consider these restrictions as too interesting now we do not present them here.

6. Infinitesimal Elastic Deformations

In this section we deal with the case of the infinitesimal elastic deformations and all their derivatives. The plastic deformations are in this section arbitrary. This assumption may be expressed as

$$x^i = \psi^i(p,t) = p^i + \eta f^i(p,t)\,, \tag{6.1}$$

where $\eta \ll 1$ is a positive constant. We obtain by derivation of this equation

$$X_k^i = \delta_k^i + \eta\frac{\partial f^i}{\partial p_k} = \delta_k^i + \eta f_k^i \tag{6.2}$$

and

$$\frac{\partial x^i}{\partial t} = \eta\frac{\partial f^i}{\partial t}\,. \tag{6.3}$$

As we have assumed that η is very small, it is a good approximation to restrict ourselves to the terms with a lower order in η.

First, for the elastic strain tensor (5.5) we obtain

$$E_{rs} = X_r^i X_s^i = \delta_{rs} + \eta(f_r^s + f_s^r) + \eta^2 f_r^i f_s^i$$

and thus

$$E_{rs} = \delta_{rs} + 2\eta\varepsilon_{rs} \tag{6.4}$$

where

$$\varepsilon_{rs} = \tfrac{1}{2}(f_r^s + f_s^r) \tag{6.5}$$

holds up to the terms of the first order in η.

For the stress tensor using the expansion with respect to η up to the first order terms we get

$$\Sigma_p^{rs} = \Sigma_p^{rs}(\boldsymbol{I}, \boldsymbol{Y}) + 2\,\eta \frac{\partial \Sigma_p^{rs}}{\partial E_{ik}}(\boldsymbol{I}, \boldsymbol{Y})\varepsilon_{ik}$$

Assuming the elastic strain tensor equals to identity, the stress tensor vanishes, it follows from the previous equation

$$\Sigma_p^{rs} = \eta c_{rsik}\varepsilon_{ik}\,, \tag{6.6}$$

where we have denoted

$$c_{rsik}(\boldsymbol{Y}) = 2\,\frac{\partial \Sigma_p^{rs}}{\partial E_{ik}}(\boldsymbol{I}, \boldsymbol{Y})\,.$$

It holds under the same approximation

$$\Sigma_p^{rs} = S_p^{rs} = T^{rs}\,, \tag{6.7}$$

where T^{rs} is the stress tensor in the Euler's coordinates. Therefore in this case we need not distinguish the stress tensor in the plastic and in the Euler's coordinates. Thus the motion equation is approximated by

$$\varrho_p v^{i\prime} = \varrho_p b^i + \frac{\partial \Sigma_p^{ik}}{\partial p^k}\,. \tag{6.8}$$

Now we consider the yield surface as the function of ε_{ik} and Y_α^i, i.e. $\varphi(\varepsilon, \boldsymbol{Y}) = 0$. By derivation we obtain the relation

$$(V\varphi) = \frac{\partial \varphi}{\partial \varepsilon_{rs}}\,\varepsilon'_{rs} + \frac{1}{2\,\eta}\frac{\partial \varphi}{\partial \varepsilon_{rs}}\left(\frac{\partial u^r}{\partial p^s} + \frac{\partial u^s}{\partial p^r}\right)\,. \tag{6.9}$$

We cannot properly use the limit $\eta \to 0$ in this equation, as we have assumed nothing about the plastic deformations, hitherto. Therefore, we cannot omit the first term with respect to the second one in this equation.

The equation (4.13) for the rate of the plastic deformations has the same form but we consider now $\boldsymbol{m}$ as a function ε_{rs} and Y_r^α. Finally, the condition (4.15) is

$$\left(Y_r^\alpha \frac{\partial \varphi}{\partial Y_s^\alpha} + \frac{1}{\eta}\frac{\partial \varphi}{\partial \varepsilon_{rs}}\right) m_s^r = 1\,, \tag{6.10}$$

where we omit the terms of the higher order in η again.

It follows

$$v^i - u^i = \eta f^{i\prime} \tag{6.11}$$

from the equation (4.19). Substituting from this equation to the motion equation (6.8), we obtain

$$\varrho_p(u^{i\prime} - b^i) + \eta\left[\varrho_p f^{i\prime\prime} - \frac{\partial}{\partial p^k}(c_{ikrs}\varepsilon_{rs})\right] = 0$$

and, it could be seemed that we can omit the second term with respect to the first one. But since we have assumed nothing about the plastic deformations, it is, like in the equation (6.9), inadmissible, in general.

7. Infinitesimal Elastic–Plastic Deformations

In this section we deal with the case, which the classic elastic–plastic theory studies, i.e. the case, when it is possible to consider both the elastic and the plastic deformations and all their derivatives as infinitesimal. The elastic displacements are given by the equation (6.1) again and similarly, the plastic displacements have the form

$$p^i = \xi^i + \eta g^i(\xi, t)\,. \tag{7.1}$$

Since we use the plastic coordinates as the independent variables p it is suitable to write the inverse equation. Neglecting the terms of higher order in η, we get

$$\xi^i = p^i - \eta g^i(p, t)\,. \tag{7.2}$$

It follows from the equations (4.16)–(4.19), (6.1) and (7.2) (up to the first order in η again)

$$Y^i_k = \delta^i_k - \eta \frac{\partial g^i}{\partial p^k} = \delta^i_k - \eta g^i_k \tag{7.3}$$

$$u^k = \eta \frac{\partial g^k}{\partial t} \tag{7.4}$$

$$\frac{\partial x^i}{\partial p^k} = \delta^i_k + \eta \frac{\partial f^i}{\partial p^k} = \delta^i_k + \eta f^i_k \tag{7.5}$$

$$v^i = \eta \left(\frac{\partial f^i}{\partial t} + \frac{\partial g^i}{\partial t} \right)\,. \tag{7.6}$$

Thus it is obvious that both u^i and v^i have the order one in η.

It follows from the continuity equation (4.6) after omitting the terms of higher order in η

$$\frac{\partial \varrho_p}{\partial t} = 0 \tag{7.7}$$

and thus in this approximation ϱ_p is a function of the coordinates p only.

The stress tensor Σ_p^{rs} is now a function of ε_{ik} and g^i_k, and hence the functions c_{rsik} are functions only g^i_k. If we restrict ourselves in the motion equation (6.8) to the terms of the lowest order in η, we obtain

$$\varrho_p \frac{\partial v^i}{\partial t} = \varrho_p b^i + \frac{\partial \Sigma_p^{ik}}{\partial p^k}\,. \tag{7.8}$$

The yield surface is given by the equation $\varphi(\varepsilon, g) = 0$ and

$$(V\varphi) = \frac{\partial \varphi}{\partial \varepsilon_{rs}} \frac{\partial}{\partial t}(\varepsilon_{rs} + \beta_{rs}) \tag{7.9}$$

holds, where we introduced the infinitesimal plastic strain tensor

$$\beta_{rs} = \tfrac{1}{2}(g^r_s + g^s_r) = \beta_{sr}\,. \tag{7.10}$$

If we introduce the infinitesimal plastic rotation tensor as

$$\omega_{rs} = \tfrac{1}{2}(g_s^r - g_r^s) = -\omega_{sr} \tag{7.11}$$

and we denote

$$U_{ik} = \tfrac{1}{2\eta}(m_k^i + m_i^k) \tag{7.12}$$

and

$$W_{ik} = \tfrac{1}{2\eta}(m_k^i - m_i^k) \tag{7.13}$$

and the dot over the letters denotes the time derivative, we obtain from the equation (4.13)

$$\dot{\beta}_{ik} = \frac{\partial\varphi}{\partial\varepsilon_{rs}}(\dot{\varepsilon}_{rs} + \dot{\beta}_{rs})\,U_{ik} \tag{7.14}$$

and

$$\dot{\omega}_{ik} = \frac{\partial\varphi}{\partial\varepsilon_{rs}}(\dot{\varepsilon}_{rs} + \dot{\beta}_{rs}\ W_{ik} \tag{7.15}$$

Finally, the condition (4.15) has the form

$$\frac{1}{\eta}\left(-\frac{\partial\varphi}{\partial g_s^r} + \frac{\partial\varphi}{\partial\varepsilon_{rs}}\right) m_s^r = -\frac{\partial\varphi}{\partial\omega_{rs}} W_{rs} + \left(-\frac{\partial\varphi}{\partial\beta_{rs}} + \frac{\partial\varphi}{\partial\varepsilon_{rs}}\right) U_{rs} = 1\,. \tag{7.16}$$

So far we restrict ourselves in this section to the general equations for the elastic–plastic deformations. Now we shortly mention the restrictions which are valid when the material has a certain symmetry mentioned in the section 5. If we assume the isotropy, the functions φ, $\boldsymbol{\Sigma}_p$ and $\boldsymbol{m}$ are dependent only on the tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\beta}$. If we assume the symmetry with respect to the all unitary transformations of the Lagrange's coordinates, these functions depend only on the $\boldsymbol{\varepsilon}$ and $\beta_1 = \mathrm{Tr}\boldsymbol{\beta}$, what is the infinitesimal plastic volume deformation.

APPENDIX

We derive in the appendix some conditions which follow for the function M_α^i from the equation (3.14). As we see later, this function, which we consider as the function of the variables F_α^i and P_α^i, satisfies except of the condition (3.7) any differential equations.

We consider a point inside the body which is with its neighbourhood in the plastic range. Thus

$$\varphi(\boldsymbol{F}, \boldsymbol{P}) = 0 \qquad \text{and} \quad (V\varphi) > 0 \tag{A.1}$$

holds in this neighbourhood. Further, we have

$$\phi = M_\rho^r \frac{\partial\varphi}{\partial P_\rho^r} = -1 \tag{A.2}$$

and

$$U_\alpha^i = (V\varphi) M_\alpha^i\,. \tag{A.3}$$

We suppose that all processes which satisfy the equations (A.1) and (A.2) are admissible in the neighbourhood of the above mentioned point. It means, it is not possible to obtain any other independent relation between $\boldsymbol{F}$ and $\boldsymbol{P}$ from the equation (3.14).

For simplicity of notation we denote

$$c^{i\kappa}_{\alpha k} = M^i_\alpha \frac{\partial \varphi}{\partial F^k_\kappa} \tag{A.4}$$

and

$$Z^\kappa_k = \frac{\partial \phi}{\partial F^k_\kappa} + c^{r\kappa}_{\rho k} \frac{\partial \phi}{\partial P^r_\rho} \,. \tag{A.5}$$

We obtain

$$\frac{\partial \phi}{\partial t} = Z^\kappa_k V^k_\kappa = 0 \tag{A.6}$$

by the derivation the equation (A.2) with respect to t. It is evident that the equation (A.2) is equivalent to the time derivative of the (A.1). Deriving the equation (A.6) with respect to ξ^τ we obtain

$$Z^\kappa_{k,\tau} V^k_\kappa + Z^\kappa_k V^k_{\kappa\tau} = 0\,. \tag{A.7}$$

The equation (3.14) is equivalent to

$$\epsilon_{\rho\sigma\tau} c^{i\kappa}_{\sigma k,\tau} V^k_\kappa + \epsilon_{\rho\sigma\tau} c^{i\kappa}_{\sigma k} V^k_{\kappa\tau} = 0\,. \tag{A.8}$$

This equation has to hold for any admissible V^k_κ and $V^k_{\kappa\tau}$. But these functions satisfy the equations (A.6) and (A.7) and therefore, they are not quite arbitrary. We use the method of the Lagrange's multipliers in order to consider them as independent and thus the terms standing before them to identify with zero. It follows from the equations (A.7) and (A.8)

$$(\epsilon_{\rho\sigma\tau} c^{i\kappa}_{\sigma k,\tau} - \lambda^{i\tau}_\rho Z^\kappa_{k,\tau}) V^k_\kappa + (\epsilon_{\rho\sigma\tau} c^{i\kappa}_{\sigma k} - \lambda^{i\tau}_\rho Z^\kappa_k) V^k_{\kappa\tau} = 0 \tag{A.9}$$

for arbitrary functions $\lambda^{i\tau}_\rho$ of the variables $\boldsymbol{F}$ and $\boldsymbol{P}$.. Since $V^k_{\kappa\tau} = V^k_{\tau\kappa}$, it follows from the equation (A.9)

$$\epsilon_{\rho\sigma\tau} c^{i\kappa}_{\sigma k} + \epsilon_{\rho\sigma\kappa} c^{i\tau}_{\sigma k} - \lambda^{i\tau}_\rho Z^\kappa_k - \lambda^{i\kappa}_\rho Z^\tau_k = 0$$

Multiplying this equation by $\epsilon_{\rho\sigma\tau}$ and summing it over β and τ we obtain

$$c^{i\kappa}_{\sigma k} = \tfrac{1}{3}\left[\epsilon_{\sigma\rho\tau}(\lambda^{i\tau}_\rho Z^\kappa_k + \lambda^{i\kappa}_\rho Z^\tau_k) + \delta^\kappa_\sigma c^{i\rho}_{\rho k}\right]\,.$$

Substituting this relation into the foregoing equation, we get

$$\left(\lambda^{i\sigma}_\rho + \lambda^{i\rho}_\sigma\right) Z^\tau_k + \left(\lambda^{i\tau}_\rho + \lambda^{i\rho}_\tau\right) Z^\sigma_k + \left(\lambda^{i\tau}_\sigma + \lambda^{i\sigma}_\tau\right) Z^\rho_k = 0$$

Under the assumption that Z_k^κ are arbitrary

$$\lambda_\tau^{i\sigma} = \epsilon_{\rho\sigma\tau} \Gamma_\rho^i \tag{A.10}$$

is obtained and thus

$$c_{\sigma k}^{i\kappa} = \Gamma_\sigma^i Z_k^\kappa + \tfrac{1}{3}\delta_\sigma^\kappa \left(c_{\rho k}^{i\rho} - \Gamma_\rho^i Z_k^\rho \right) \tag{A.11}$$

holds. Substituting Z_k^κ from the (A.5) into the (A.11), we obtain

$$\left(c_{\tau k}^{r\rho} S_{\alpha r}^{i\tau} - \Gamma_\alpha^i \frac{\partial \phi}{\partial F_\rho^k} \right) R_{\sigma\rho}^{\kappa\alpha} = 0\,, \tag{A.12}$$

where

$$S_{\sigma r}^{i\tau} = \delta_r^i \delta_\sigma^\tau - \Gamma_\sigma^i \frac{\partial \phi}{\partial P_\tau^r}$$

and

$$R_{\sigma\rho}^{\kappa\alpha} = \delta_\rho^\kappa \delta_\sigma^\alpha - \tfrac{1}{3}\delta_\sigma^\kappa \delta_\rho^\alpha$$

It follows from the equation (A.12)

$$c_{\tau k}^{r\rho} S_{\alpha r}^{i\tau} = \Gamma_\alpha^i \frac{\partial \phi}{\partial F_\rho^k} + T_k^i \delta_\alpha^\rho$$

and substituting $S_{\alpha r}^{i\tau}$ into this equation, we obtain

$$c_{\sigma k}^{i\kappa} = \Gamma_\sigma^i Z_k^\kappa + T_k^i \delta_\sigma^\kappa\,. \tag{A.13}$$

Since we have assumed that Z_k^κ are arbitrary, we may consider

$$Z^2 = Z_k^\kappa Z_k^\kappa \neq 0\,.$$

In this case, it follows from the relation (A.13)

$$\Gamma_\sigma^i = Z^{-2} \left[c_{\sigma k}^{i\kappa} Z_k^\kappa - T_k^i Z_k^\sigma \right]\,.$$

Remark. On the other hand, $Z = 0$ implies $Z_k^\kappa = 0$ for any k and κ and we obtain the additional equation for $\boldsymbol{F}$ and $\boldsymbol{P}$. As the condition (A.6) is satisfied identically in this case, we must proceed in other way.

In this way we annul the terms with $V_{\kappa\tau}^k$ in the equation (A.9). Thus, it remains to annul the first term. Substituting of $c_{\sigma k}^{i\kappa}$ from the equation (A.12) and $\lambda\rho^{i\tau}$ from (A.10), we obtain the relation

$$\epsilon_{\rho\sigma\tau} \left(\Gamma_{\sigma,\tau} Z_k^\kappa + \delta_\sigma^\kappa T_{k,\tau}^i \right) V_\kappa^k = 0\,. \tag{A.14}$$

Also in this equation V_κ^k are not independent, because the equation (A.6) holds. Using again the trick with the Lagrange's multipliers, we obtain

$$\epsilon_{\rho\sigma\tau}\left(\Gamma^i_{\sigma,\tau} Z^\kappa_k + \delta^\kappa_\sigma T^i_{k,\tau}\right) - \mu^i_\rho Z^\kappa_k = 0\,, \tag{A.15}$$

where the corresponding Lagrange's multipliers are μ^i_ρ.

Now, we finally specify the outline derivation with respect to ξ^τ. Carrying out this derivations, we obtain

$$\epsilon_{\rho\sigma\tau}\left(\frac{\partial\Gamma^i_\sigma}{\partial F^a_\alpha} Z^\kappa_k + \delta^\kappa_\sigma\frac{\partial T^i_k}{\partial F^a_\alpha}\right) F^a_{\alpha\tau} + \epsilon_{\rho\sigma\tau}\left(\frac{\partial\Gamma^i_\sigma}{\partial P^a_\alpha} Z^\kappa_k + \delta^\kappa_\sigma\frac{\partial T^i_k}{\partial P^a_\alpha}\right) P^a_{\alpha\tau} - \mu^i_\rho Z^\kappa_k = 0\,.$$

But the functions $F^a_{\alpha\tau}$ and $P^a_{\alpha\tau}$ are bound by the equations which we get by derivation of the equations (A.1) and (A.2) with respect to ξ^τ, i.e.

$$\frac{\partial\varphi}{\partial F^a_\alpha} F^a_{\alpha\tau} + \frac{\partial\varphi}{\partial P^a_\alpha} P^a_{\alpha\tau} = 0$$

and

$$\frac{\partial\phi}{\partial F^a_\alpha} F^a_{\alpha\tau} + \frac{\partial\phi}{\partial P^a_\alpha} P^a_{\alpha\tau} = 0\,.$$

Casting the Lagrange's multiplier μ^i_ρ in the form

$$\mu^i_\rho = \omega^{i\alpha\tau}_{\rho a} F^a_{\alpha\tau} + \varpi^{i\alpha\tau}_{\rho a} P^a_{\alpha\tau}$$

we have

$$\left[\epsilon_{\rho\sigma\tau}\left(\frac{\partial\Gamma^i_\sigma}{\partial F^a_\alpha} Z^\kappa_k + \delta^\kappa_\sigma\frac{\partial T^i_k}{\partial F^a_\alpha}\right) - \omega^{i\alpha\tau}_{\rho a} Z^\kappa_k - \nu^{i\kappa\tau}_{\rho k}\frac{\partial\varphi}{\partial F^a_\alpha} - \varsigma^{i\kappa\tau}_{\rho k}\frac{\partial\phi}{\partial F^a_\alpha}\right] F^a_{\alpha\tau} = 0$$

$$\left[\epsilon_{\rho\sigma\tau}\left(\frac{\partial\Gamma^i_\sigma}{\partial P^a_\alpha} Z^\kappa_k + \delta^\kappa_\sigma\frac{\partial T^i_k}{\partial P^a_\alpha}\right) - \varpi^{i\alpha\tau}_{\rho a} Z^\kappa_k - \nu^{i\kappa\tau}_{\rho k}\frac{\partial\varphi}{\partial P^a_\alpha} - \varsigma^{i\kappa\tau}_{\rho k}\frac{\partial\phi}{\partial P^a_\alpha}\right] P^a_{\alpha\tau} = 0$$

where $\nu^{i\kappa\tau}_{\rho a}$ and $\varsigma^{i\kappa\tau}_{\rho k}$ are any other Lagrange's multipliers. Since $F^a_{\alpha\tau} = F_{\tau\alpha}$ and $P^a_{\alpha\tau} = P^a_{\tau\alpha}$ hold, it follows from these equations

$$\begin{aligned}&\left[\epsilon_{\rho\sigma\tau}\left(\frac{\partial\Gamma^i_\sigma}{\partial F^a_\alpha} Z^\kappa_k + \delta^\kappa_\sigma\frac{\partial T^i_k}{\partial F^a_\alpha}\right) - \omega^{i\alpha\tau}_{\rho a} Z^\kappa_k - \nu^{i\kappa\tau}_{\rho k}\frac{\partial\varphi}{\partial F^a_\alpha} - \varsigma^{i\kappa\tau}_{\rho k}\frac{\partial\phi}{\partial F^a_\alpha}\right] +\\ &+\left[\epsilon_{\rho\sigma\alpha}\left(\frac{\partial\Gamma^i_\sigma}{\partial F^a_\tau} Z^\kappa_k + \delta^\kappa_\sigma\frac{\partial T^i_k}{\partial F^a_\tau}\right) - \omega^{i\tau\alpha}_{\rho a} Z^\kappa_k - \nu^{i\kappa\alpha}_{\rho k}\frac{\partial\varphi}{\partial F^a_\tau} - \varsigma^{i\kappa\alpha}_{\rho k}\frac{\partial\phi}{\partial F^a_\tau}\right] = 0\end{aligned} \tag{A.16}$$

and

$$\begin{aligned}&\left[\epsilon_{\rho\sigma\tau}\left(\frac{\partial\Gamma^i_\sigma}{\partial P^a_\alpha} Z^\kappa_k + \delta^\kappa_\sigma\frac{\partial T^i_k}{\partial P^a_\alpha}\right) - \varpi^{i\alpha\tau}_{\rho a} Z^\kappa_k - \nu^{i\kappa\tau}_{\rho k}\frac{\partial\varphi}{\partial P^a_\alpha} - \varsigma^{i\kappa\tau}_{\rho k}\frac{\partial\phi}{\partial P^a_\alpha}\right] +\\ &+\left[\epsilon_{\rho\sigma\alpha}\left(\frac{\partial\Gamma^i_\sigma}{\partial P^a_\tau} Z^\kappa_k + \delta^\kappa_\sigma\frac{\partial T^i_k}{\partial P^a_\tau}\right) - \varpi^{i\tau\alpha}_{\rho a} Z^\kappa_k - \nu^{i\kappa\alpha}_{\rho k}\frac{\partial\varphi}{\partial P^a_\tau} - \varsigma^{i\kappa\alpha}_{\rho k}\frac{\partial\phi}{\partial P^a_\tau}\right] = 0\,.\end{aligned} \tag{A.17}$$

Thus, M^i_α have to satisfy, except of the relation (A.2), the system of the partial differential equations (A.4), (A.5), (A.14), (A.16) an (A.17) in order to the relation (3.14) holds. Or, since (A.4), (A.5) and (A.14) are, in fact, the definitions, it is possible to substitute from these equations to the (A.16) and (A.17) and in such a way to obtain the system of the partial differential equations for M^i_α, which depends on the functions $\boldsymbol{T}$, ω, ϖ, $\boldsymbol{\nu}$ a ς. But, there is a question, whether it is possible to choose these functions in such a way the resulting system of the partial differential equations has a solution. It should be of an interest to solve this problem for a given function φ at least in any more simple cases when we suppose any symmetry mentioned in the section 5 or for the infinitesimal elastic–plastic deformations studied in the section 7.

Conclusion

We are fully aware of the fact that the theory presented in this paper has no experimental verification. The theoretical reasons, which led me to the creation of this theory, are:

- author's opinion that the elastic–plastic material has the property that, if it is in the equilibrium it is always possible to find any external force field to deform this material into the state in which the stress vanishes, exclusively by the elastic way;
- from this assumption following well-defined decomposition of the total deformations to the elastic and plastic parts,
- easy definition of the elastic–plastic theory for the finite deformations and with this connected the description of the material, in which the elastic deformations are infinitesimal and the plastic ones are finite, e.g. plastic flow.

The consequence of this theory is the restriction of the constitutive relations to the rate of the plastic deformations, which are defined in the elastic–plastic theory of the infinitesimal deformations by an associative or a non-associative law of flow.

We think that the basic problem in the theory presented here it is the study of the system of the partial differential equations (A.16) and (A.17) for a given yield surface $\varphi(\boldsymbol{F}, \boldsymbol{P}) = 0$.

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