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THE CALCULATION OF HYDRAULIC CONDUCTIVITY OF SANDS ACCORDING TO THEIR GRANULARITY

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ABSTRACT. The hydraulic conductivity of sands depends mainly on their granulometric composition, which heterogeneity is expressed with the ratio $U = d_{60}/d_{10}$, where d_{60}, (d_{10}) is the grain size belonging to 60 % (10 %) of the undersizes. The representative size of grains is being defined in different ways, often, it is the value d_{10}, sometimes it is d_w, which in substance is the size of grains, which hydraulic resistance would be equal to resistance of the equal number of different size grains of given soil.

In this study, a proposition is given to describe the granulometrics of sands with the characteristics of Gauss logarithmic – normal distribution LN(μ, σ^2), and so, with the factors μ, σ, which would also be the basis for determining of the coefficient of hydraulic conductivity.

1. INTRODUCTION

All the formulas for calculation of hydraulic conductivity factor k have the basic form of

$$k = C \cdot d^2 \qquad (\mathrm{ms}^{-1}) \tag{1}$$

The size d of a grain is established by the granularity analysis.

The granularity composition of loose materials was formerly described by using Weibull distribution (Likeš and Machek, 1982), which was already introduced in the field of minerals processing. It became obvious though, that the granularity of sands can be expressed much more exactly by using logarithmic–normal distribution than by using the Weibull distribution. If we approximate the analysis of sand granularity with the logarithmic-normal distribution of LN(μ, σ^2) the idea offers itself to use the parameters μ, σ^2 of this distribution to determine the values of C and d, and thus also to calculate the coefficient of hydraulic conductivity.

The logarithmic–normal distribution was accepted as the most suitable for the description of granular, porous and also fissure environment at the XXIV. International Hydrogeological Congress in Oslo, 1993; (Hydrogeology of Hard Rocks).

For this study, a series of experiments was conducted, which were to verify the validity of the formula (1), and the influence of tortuosity on a hydraulic resistance of granular environment.

2. The Static Moments of Logarithmic – Normal Distribution

The median value of r–th static moment $E(r)$ of this distribution is

$$E(r) = \int_0^\infty x^r f(x)\,\mathrm{d}x\,; \quad f(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right] \tag{2}$$

It is possible to deduce (Likeš and Machek, 1982) that

$$E(r) = \exp\left(r\mu_r - \frac{r^2}{2}\,\sigma^2\right) \tag{3}$$

$$\mu_r = \mu + r\sigma^2 \tag{4}$$

The factors μ_3, σ are established by representation of the dependence of undersizes increase on the size of grains in the probabilistic net for LN(μ, σ^2) of distribution (Busch and Luckner, 1972; Likeš and Machek, 1983). In the majority of cases, it is possible to represent this dependence as a straight line. (straight line p in the Fig. 1).

From the position and inclination of the line it is already possible to determine both factors (Likeš and Machek, 1983).

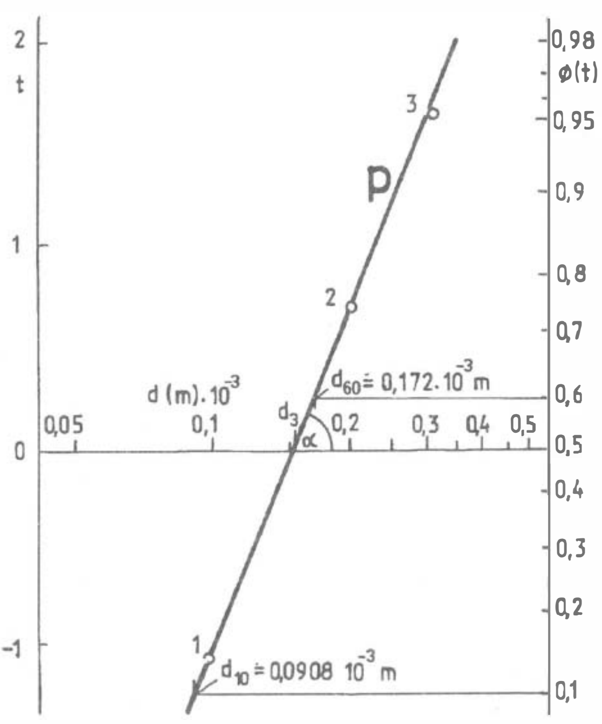

FIG. 1. The work nomogram

The volume and mass of grains are proportional to the cube of the size of the grain. If $f(x)\,dx$ is frequency of grains of the size x, then the expression $x^3 f(x)\,dx$ is proportional to volume (or mass) of grains of this size. So, the undersizes are expressed by the third static moment, from which the factors μ_3, σ are determined. (σ is identical for all moments)

The factor μ_1 for the first static moment, which we will use to determine the hydraulic resistance of grains, is calculated from the expression (4). On Fig. 1, the results of screen analysis (Table 1) are shown as points 1,2,3. In the logarithmic–normal probabilistic net, the dependence of undersizes $\phi(t)$ on the size x of grains is represented by the straight line p, which crosses the axis d in the point d_3, under the angle of α. The factor $t = (\ln x - \mu_3)/\sigma$.

TABLE 1. The results of sieve analysis of a sand

measurement No	grains size $[10^{-3}\,\mathrm{m}]$	undersize [kg]	undersize (relative) $\phi(t)$
1	0.1	0.0405	0.1406
2	0.2	0.2175	0.7549
3	0.315	0.2742	0.9517
4	∞	0.2881	1.0000

The factors of distribution are calculated from formulas[1]

$$\begin{aligned} \mu_3 &= \ln d_3 \\ \sigma &\doteq \ln U/1.535 \\ U &= d_{60}/d_{10} \end{aligned} \tag{5}$$

where d_{60} (d_{10}) is the grain size belonging to 60 % (10 %) of undersizes.

In the (Likeš and Machek, 1982), σ is determined from the inclination od line p, using the relation

$$\frac{1}{\sigma} = \operatorname{tg}\alpha\,.$$

In the equation (5), $\operatorname{tg}\alpha$ is determined by the traditionally used criterion of size heterogeneity U (in German Ungleichformigkeit). The inclination $\operatorname{tg}\alpha = \big(t_{(60)} - t_{(10)}\big)/\big(\ln d_{60} - \ln d_{10}\big)$, where $t(60)$, ($t(10)$) are parameters of distribution function $\phi(t)$, when $\phi(t) = 0.6$; $\phi(t) = 0.1$. In (Hátle and Likeš, 1974), on page 441, 442 is used symbol u instead of t. From the tables of normal distribution we find out, that $t(60) - t(10)$ equals approximately 1.535, and this value was used to deduce the second equation from (5).

The median value of the third static moment $E(3)$ determines the volume of the medium–sized grain of cube shape; should we suppose the sphere shaped grain, the

[1] The graphical methods of interpretation of empirical and probabilistic functions are described in details in (Likeš and Machek, 1983) on the page 113–124.

volume is $(\pi/6) \cdot E(3)$. The number of grains n in the unit of sample volume is determined by

$$n = \frac{6(1-m)}{\pi \cdot E(3)}. \tag{6}$$

On the examined sample, a porosity of $m = 0.4$ was detected. For the characteristics of the sample, given in Tab. 1 and Fig. 1, we will deduce the following from the equation (5): $\mu_3 = -8.7721$ a $\sigma = 0.416$, from formula (3) is $E(3) = 1.709 \cdot 10^{-12}$ and from the formula (6) $n = 670 \cdot 10^9$, the count of grains in $1\,\mathrm{m}^3$ of soil.

The dependence between the porosity and the number of grains in the unit of volume is therefore simple. The dependence between the porosity and flow area of gaps between grains is more complex though. This dependence can be expressed only for simple geometric arrangement of spherical grains, and that is, for square and hexagonal arrangement. The formulas of porosity for both arrangements are given in (Busch and Luckner, 1972) and (Polubarinova-Kochina, 1962). To calculate a hydraulic resistance, it is necessary to know even the minimum flow profile and its hydraulic diameter.

The minimum flow profile for both arrangements of spherical grains is given in the Figs. 2a and 2b. The hydraulic diameter d_0 (Maštovský, 1956) is defined by the formula $d_0 = 4F/O$, where O is a wet circumference of profile F. Both these quantities can be expressed in a dimensionless form; for a profile to the ratio will be $f = F/F_0$ (the meaning of F_0 is shown on Figs .2a and b). For a hydraulic diameter, to will be a ratio (d_0/d).

The characteristics of f, (d_0/d) are given on Fig. 2c – their course between the points a), b), we suppose as a line.

3. The Flow of Water in Gaps between Grains

Provided the velocity of the flow is not high – approximately up to Re < 5, the force R, applied to grains, is given by the formula

$$R = \frac{3\pi\nu\rho v d}{(d_0/d)}. \qquad \text{(N)} \tag{7}$$

This formula is known as Stokes formula for $d_0 = d$. A dimensionless quantity f, specified in the previous paragraph, indicates the size of flow profile in the element with a unit edge.

The gravity of water flowing through this unit element therefore is $\rho g f$. If there are n grains in the element, the force of flowing water acting on them is nR. In the static state, both forces are at equilibrium, $\rho g f = nR$.

The velocity v in the (7) means the velocity in the flow profile f; in fact it is a filtration speed k, related to profile f, in other words, $v = k/f$. A relative hydraulic diameter (d_0/d) in the (7) is for a given arrangement of grains constant, independent of the size of grain.

Calculating the total force nR, it is necessary to substitute a median size of grain d in the formula (7). Provided the median volume of grain is given by the median

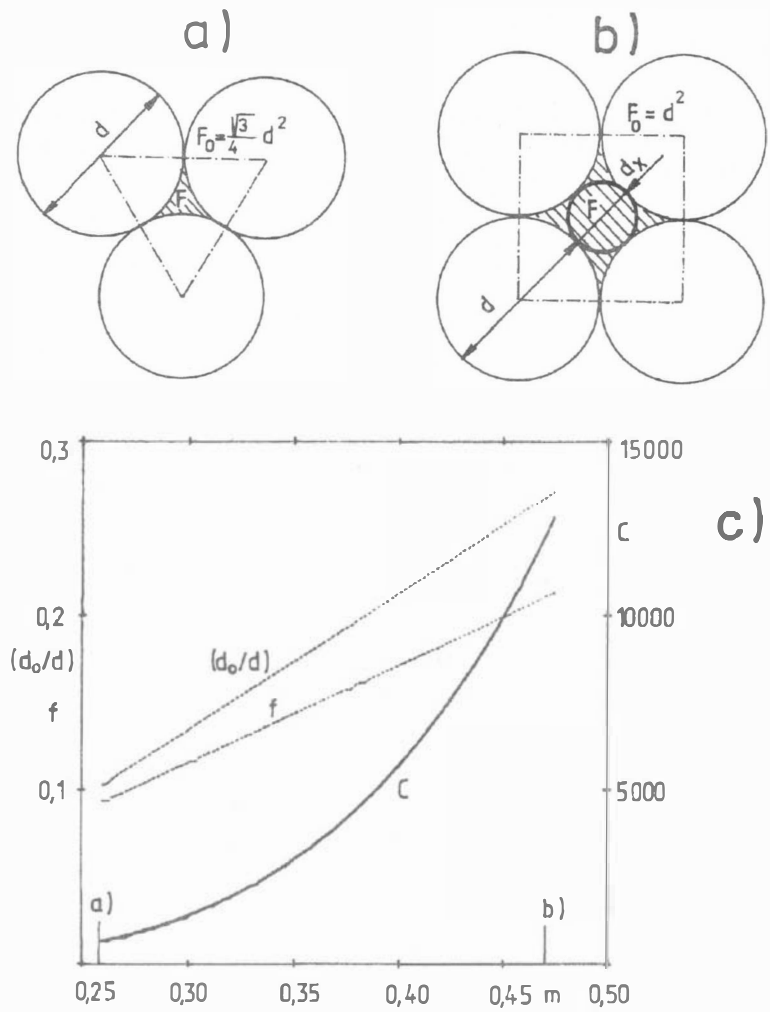

FIG. 2. The diagram of spherical grains arrangements a) hexagonal, b) cubic; c) of dependence of f, (d_0/d), C on the porosity m

value of third static moment $E(3)$, the median size of grain is given by the median value of the first static moment $E(1)$. From the given relations, we will determine the filtration velocity k

$$k = \frac{g f^2 \, (d_0/d)}{n 3 \pi \nu E(1)} \cdot \qquad (\mathrm{ms}^{-1}) \tag{8}$$

We know that k is also a constant of proportionality between gradient of hydraulic potential and velocity of flow, and so, it indicates the value of the coefficient of hydraulic conductivity.

From the curve of granularity, see Fig. 1, we determine factors d_3, d_{60}, d_{10}. From equations (5) we calculate μ_3, σ, by means of (4) we determine μ_1, and from (3) the median value $E(1)$. We can express the coefficient of hydraulic conductivity also by d_3 and σ, which we determine from curves of granularity. Substituting from (3) to (7) into (8), and consulting results of experiments, we deduce this formula for calculating the coefficient of hydraulic conductivity:

$$k = Cf(\sigma)\exp\left(-2\sigma^2\right) d_3^2\,, \qquad (\mathrm{ms}^{-1}) \tag{9}$$

where

$$C = \frac{gf^2\,(d_0/d)}{18\nu(1-m)}\,. \qquad (\mathrm{m}^{-1}\mathrm{s}^{-1})$$

The factor $f(\sigma)$ will be determined from Fig. 4 – see the following chapter "Description of experiments". We know that m, f, (d_0/d) we can calculate only for hexagonal and cube arrangements of spherical grains, by which these variables reach extreme values.

In the hexagonal arrangement, porosity, flow profile and its hydraulic diameter are at the minimum, in the cube arrangement, these factors are biggest possible to attain with spherical, mutually touching grains.

Directly, it is possible to determine only the porosity. We know for sure, that the flow profile, and also hydraulic profile, grow as the porosity grows. Supposing the growth is linear, for given porosity m, we can easily determine not only f and (d_0/d), but also the constant C in the equation (9); the dependence of these variables on porosity is shown in Fig. 2c.

The example. At the sample, which characteristics are $d_3 = 0.155 \cdot 10^{-3}$ m, $\sigma = 0.416$ – see Table 2, measurement number 3 – the porosity was determined to be $m = 0.39$. For this porosity we determine relative flow profile $f \approx 0.172$ from the Fig. 2c; relative hydraulic diameter $(d_0/d) \approx 0.213$. The gravitational acceleration $g = 9.81\ \mathrm{ms}^{-2}$, kinematic viscosity of water ν equals $10^{-6}\ \mathrm{m}^2\mathrm{s}^{-1}$. From the diagram on the Fig. 4 we determine for $\sigma = 0.416$ the factor $f(\sigma) \doteq 0.48$. From the equation (9) we calculate the coefficient of hydraulic conductivity $k \approx 4.1 \cdot 10^{-5}\ \mathrm{ms}^{-1}$, that is, about 5 % higher than experimentally determined $k = 3.95 \cdot 10^{-5}\ \mathrm{ms}^{-1}$.

TABLE 2. The results of measurements of coefficient of hydraulic conductivity

measurement No	size d_3 $[10^{-3}]$ m	σ-factor of heterogeneity	coefficient of hydraulic conductivity k $[\mathrm{ms}^{-1}]$
1	0.10	0.39	$1.95 \cdot 10^{-5}$
2	0.40	0.37	$3.02 \cdot 10^{-4}$
3	0.155	0.416	$3.95 \cdot 10^{-5}$
4	0.35	0.43	$1.58 \cdot 10^{-4}$
A	0.225	0.00	$2.97 \cdot 10^{-4}$
B	0.45	0.00	$1.16 \cdot 10^{-3}$

4. The Description and Results of Experiments

The hydraulic conductivity was determined in samples of volume of $1\,\mathrm{dm}^3$, through which $2-5\,\mathrm{dm}^3$ of water with the temperature of $18\,°\mathrm{C}$ was flowing. In the Table 2, there are several results given, from which two, marked A and B, are especially significant.

The experiment A: the sample of granularity of 0.225 mm was attained by sifting of sand through the sieve with holes of 0.215 and 0.235 mm. Supposing that the growth of undersizes between the granularity of 0.215 and 0.235 mm is linear, the d_{10} then equals approx 0.217 mm, $d_{60} = 0.227$, and from the formulas (5), we will calculate that the size heterogeneity of this formula must be $\sigma < 0.03$. Experimentally, it was proved that the influence of size heterogeneity appears only from $\sigma > 0.2$ and thus, it is possible to consider this formula to be size–conforming.

Experiment B: As a sample, a ballotine with granularity of 0.45 mm was used. Ballotine grains were used because they are manufactured by a special technology, aimed at keeping the given size and exact spherical form of grains. (Ballotine is a commercial name for a substance, which looks like a dull glass). The capillary elevation of ballotine is the same as that of sand, as was verified by laboratory measurement. The results of experiments showed that the formula for the coefficient of hydraulic conductivity $k = C\exp\left(-2\sigma^2\right) d_3^2$ is valid only for cases when the size heterogeneity is $\sigma = 0$, resp. $\sigma < 0.2$. Results from Table 2, shown at Fig. 3, demonstrate, though, that the values of coefficient k, acquired by measurements No 1 to 4, are much smaller than expected.

It appeared that the size heterogeneity has much greater influence on the decrease of flow, than the one expressed by $\exp\left(-2\sigma^2\right)$, in which case the points 1 to 4 would have to be on the line $\sigma = 0.4$. The size heterogeneity can cause significant decrease of flow profile by small grains filling gaps between the big grains. Fig. 2b shows the case of a spherical grain of diameter d_x. At a cubic arrangement, d_x is approx. $0.414\,d$ (exactly $d_x = \left(\sqrt{2}-1\right) d$), and this grain will decrease the flow profile by 63 %.

The probability of this decrease depends obviously on the mutual ratio of numbers of grains of sizes d and d_x, and this ration depends on the size heterogeneity σ. If the $\sigma = 0$, then the sample contains only the grains of size d and none else. With the increase of σ, the number of grains of size d_x slowly increases; by which also increases the probability, that they will fill gaps between grains d. When the heterogeneity σ crosses certain level,though, this probability will decrease again. That is because with increasing σ, the number of grains of different sizes will grow too, and so, the probability of d and d_x grains occurrence also decreases. Based on this hypothesis and on results of measurements, a formula for determining of decrease coefficient of flow profile was deduced as:

$$f(\sigma) = \left[1 - (3/\sigma^2)\cdot\phi(-1/\sigma)\right]^2 , \tag{10}$$

where $\phi(-1/\sigma)$ is standardized distribution function of normal distribution,

$$\phi(-1/\sigma) = \int_{-\infty}^{-1/\sigma} \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}} .$$

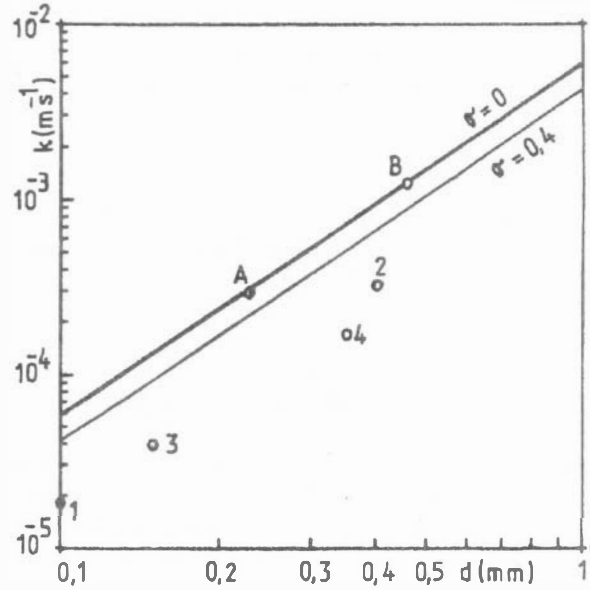

FIG. 3. The results of measurements of coefficient of hydraulic conductivity

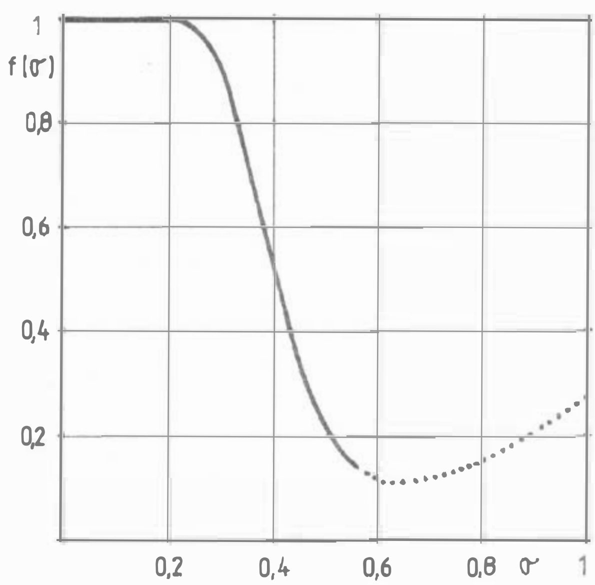

FIG. 4. The factor of flow profile decrease in relation to the size heterogeneity

This dependence (10) was being verified for $\sigma < 0.5$ only, that is why its further course from $\sigma > 0.5$ is shown as dotted in the Fig. 4.

5. Summary

The measurements of flow in samples of pure sand and little ballotine balls of granularity between 0.1 and 0.45 verified, that the hydraulic conductivity increases with the square of the median size of grain, be this median size defined either by specific surface (Hálek and Švec, 1973; Langguth and Voigt, 1980), or by hydraulic diameter, as is done in this study.

Two experiments (A, B – see Fig. 3, resp. Table 2), were aimed at determining the influence of tortuosity on the hydraulic conductivity. Each of these two experiments' samples contained grains of equal size, sample A grains of size 0.225 ± 0.01 mm, sample B contained ballotine little balls with diameter of 0.450 ± 0.005 mm. The dimensional heterogeneity and the porosity of both samples were equal, $\sigma = 0$, $m = 0.4$.

In such case, the formula (9) says that $C = k/d_3^2$; for the sample A, $C \approx 5867$, for the sample B, $C \approx 5728$.

The influence of tortuosity (see for example (Mucha and Šestakov, 1987, page 27) would be expressed by the factor 1.024 if we supposed that tortuosity at flow through sand is greater than that through ballotine balls. The factor 1.024 is too small to show the influence of tortuosity, which is distinctly different at sand than at ballotine balls.

This experiment only verified the correctness of presumption that the influence of tortuosity appears with higher velocities, when the hydraulic resistance is not determined only by viscosity, but also by forces created by changes of flow direction.

The dependence between the grain size and undersizes is mostly being demonstrated in a probabilistic, as a rule log-normal net. That is why this study too, describes the method of how to define a nominal size of grain and also the size heterogeneity with the help of standard characteristics σ, μ of log-normal distribution.

The Symbol List

k (ms^{-1})	– the coefficient of hydraulic conductivity (filtration velocity)
d (m)	– the size of grain
f	– the minimum specific flow profile in gaps between grains
ρ (kgm^{-3})	– density of water
g (ms^{-2})	– the gravitational acceleration
ν (m^2s^{-1})	– the kinematic viscosity of water
v (ms^{-1})	– the velocity of flow in the minimum flow profile between grains
m	– porosity

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