DIRECT MODELING OF THE GRAVITATIONAL FIELD USING HARMONIC SERIES

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ABSTRACT

During the General Assembly of the European Geosciences Union in April 2008, the new Earth Gravitational Model 2008 (EGM08) was released with fully-normalized coefficients in the spherical harmonic expansion of the Earth's gravitational potential complete to degree and order 2159. EGM08 is based on inverse modeling methods that rely on data observed both on the Earth's surface and in space. Forward modeling equations based on Newtonian integrals can be converted into series forms that are compatible with the spherical harmonic description of the geopotential. Namely gravitational potentials of ocean water (fluid masses below the geoid) and topographical masses (solid masses above the geoid) can be formulated and evaluated numerically through spherical harmonic expansions. The potential constituents as well as their radial derivatives can be used for a step known in geodesy and geophysics as gravity field reduction or stripping. Reducing EGM08 for these constituents can help to analyze the internal structure of the Earth (geophysics) as well as to derive the Earth's gravitational field harmonic outside the geoid (geodesy).

KEYWORDS: Earth Gravitational Model, gravity reduction and stripping, gravitational potential, ocean water, spherical harmonic series, topography

1. INTRODUCTION

Geodesy is a branch of applied science that deals with measuring and representing Earth's size and shape, external gravitational field, rotation and orientation (including temporal variations of those quantities with time). The gravitational field is a vector field that can be described by a scalar function of position and time called the gravitational potential (shortly geopotential). Values of the geopotential are generally considered to be unobservable (geopotential differences are obtainable through combined leveling and gravimetry), however, various functionals of the geopotential can be measured such as directional derivatives (gravimetry and gradiometry), deflections of the vertical (astronomic leveling) and sea surface heights (satellite altimetry). From these observables, local/global gravitational models are derived either in a form of a digital gravitational model (discrete values of the geopotential over some reference surface) or a set of numerical coefficients in a harmonic expansion of the geopotential.

Spaceborne observables of the gravitational field became widely available in recent years. Since the launch of the first gravity-dedicated satellite CHAMP in 2000, new global Earth gravitational models based solely on spaceborne data have been derived. Spaceborne data have some disadvantages that are related mainly to the attenuation of the gravitational field with altitude, however, their almost worldwide coverage results in large improvements of harmonic models of the gravitational field. Both the accuracy and resolution (maximum degree and order of the harmonic expansion) have been largely improved. Based on data from the GRACE mission, new gravitational models can be derived up to degree 180. From the GOCE mission launched in 2009, further improvements are expected (resolution up to degree 250). At the same time, new combined models based on satellite-only harmonic models and ground gravity data have been derived. The new Earth Gravitational Model 2008 (EGM08) computed by the US NGA (Pavlis et al., 2008) contains coefficients of the spherical harmonic series complete up to degree and order 2159 that corresponds to the equiangular resolution of 5 arcmin.

Although the Earth's gravitational field varies in time, EGM08 represents the static gravitational field generated by all masses of the Earth. The mass density distribution of the Earth is not homogeneous since the Earth consists of the atmosphere, hydrosphere (water in liquid and solid state), and various chemical elements forming rocks (again in liquid or solid state). Some of these mass constituents are known relatively well in terms of their 3-D mass density distribution. Especially, atmosphere, ocean and lake water, glaciers, topography and upper mantle layers can be approximated relatively well by some suitable mass density distributions.

In geodesy and geophysics, gravitational fields of selected masses are modeled by using different techniques (they are usually referred to as forward modeling techniques). Gravitating masses are being decomposed into simple volume elements, for which analytical formulas of their gravitational potentials can be derived, and the total gravitational field is computed through the well known superpositioning principle of potential theory. Different techniques for simple volume elements (tesseroids, prisms) and mass density distributions (homogeneous, linearly-varying) have been developed in time (e.g., Mader, 1951; Anderson, 1976). Alternatively, the external gravitational potential can be modeled by harmonic series expansions (e.g., Rummel et al., 1988). In this case values of harmonic coefficients must be solved for based on geometry and density distribution of gravitating masses. In this contribution, this technique is discussed and applied for selected masses with known geometry and simple mass density distribution. The harmonic models may be applied for various geodetic, geodynamic and geophysical applications and for studying spectral properties of the Earth's gravitational field.

2. SPHERICAL HARMONIC REPRESENTATION OF THE GEOPOTENTIAL

Given a geocentric spherical coordinate system with the radius *r*, co-latitude $0 \le \theta \le \pi$, and longitude $0 \le \lambda < 2\pi$, the external geopotential *V* (m²·s⁻²) can be defined by the well known Newtonian volume integral (MacMillan, 1958)

$$V(r,\Omega) = G \int_{\Theta} \int_{\xi=0}^{r(\Omega)} \rho(\xi,\Omega') L^{1}(r,\Omega,\xi,\Omega') \xi^{2} d\xi d\Omega'$$
(1)

where the following abbreviated notation was used

$$\int_{\Theta} d\Omega' = \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} \sin \theta' \, \mathrm{d}\theta' \, \mathrm{d}\lambda' \, .$$

G stands for the universal gravitational constant $(m^3 \cdot kg^{-1} \cdot s^{-2})$, ρ is the 3-D mass density function $(kg \cdot m^3)$. The geocentric position of the computation point in Eq. (1) is given by the geocentric radius *r* at the geocentric direction $\Omega = (\theta, \lambda)$ and the parameter $\Theta = \langle 0, \pi \rangle \times \langle 0, 2\pi \rangle$ stands for the full solid angle. The integration kernel L^{-1} is the inverse Euclidean distance between the computation and integration points that is expandable into a series of spherical harmonics (Heiskanen and Moritz, 1967)

$$L^{-1}(r,\Omega,\xi,\Omega') = \frac{1}{r} \sum_{n,m} \left(\frac{\xi}{r}\right)^n \frac{1}{2n+1} Y_{n,m}(\Omega) Y^*_{n,m}(\Omega') ,$$

$$r \ge \xi .$$
(2)

Another abbreviated notation was introduced as follows:

$$\sum_{n,m} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} .$$

Y and *Y*^{*} in Eq. (2) are spherical harmonics and their complex conjugates, respectively (Abramowitz and Stegun, 1972). The series in Eq. (2) is uniformly convergent if $r \ge \zeta$.

The external geopotential can be represented by the convergent spherical harmonic series with numerical coefficients $V_{n,m}$, e.g., (Torge, 2001),

$$V(r,\Omega) = \frac{GM}{R} \sum_{n,m} \left(\frac{R}{r}\right)^{n+1} V_{n,m} Y_{n,m}(\Omega) , r \ge R ,$$
(3)

with mass *M* and mean radius *R* of the Earth. Values of the coefficients $V_{n,m}$ for degree *n* and order *m* up to 2159 (selected non-zero coefficients for degrees up to 2190) can be found in EGM08. Then the series in Eq. (3) is finite and values of the gravitational potential *V* with the equiangular resolution of 5 arcmin can be synthesized anywhere outside the sphere of radius *R* (the series can also be used inside the sphere as long as we stay outside the gravitating masses).

Based on forward modeling the gravitational potential of topographical masses and ocean water will be derived in terms of the spherical harmonic series compatible with that in Eq. (3). Topographical masses are usually defined as solid Earth's masses bounded by the mean sea level (geoid) and the surface of the Earth. Obviously these masses are defined only over continents and islands. They play a very important role both in geodesy and geophysics. Ground gravity observations collected for decades can closely be related to gravitation of topographical masses since they surround ground gravity stations. Irregularities in their geometry and mass variations largely affect observed gravity data. For this reason gravity data are reduced for the gravitational effect of surrounding topographical masses (gravity reduction). Ground gravity data, especially those close to coasts, are also largely affected by sea water. In order to smooth more observed gravity, data should further be reduced for the density contrast between the mean mantle density and sea water density. This operation is usually referred to in geophysics as gravity stripping.

To describe topography and ocean water geometrically, we define two closed smooth and simply connected surfaces: solid Earth's surface r_t including topography (*hypsometry*) and ocean bottoms (*bathymetry*), and the geoid r_g representing the mean sea level. The surface bounding the solid Earth is described by a Global Topographical Model (GTM) that contains numerical coefficients $H_{n,m}$

$$r_{t}(\Omega) = R + H(\Omega) \approx R + \sum_{n,m} H_{n,m} Y_{n,m}(\Omega).$$
(4)

Harmonic coefficients $H_{n,m}$ of the height/depth function H up to degree and order 2159 were released

with EGM08. The mean sea level or the geoid can approximately be estimated from EGM08

$$r_{g}\left(\Omega\right) = R + N\left(\Omega\right) \approx R + R \sum_{n,m} N_{n,m} Y_{n,m}\left(\Omega\right).$$
 (5)

Spherical harmonic coefficients $N_{n,m}$ refer to geometric deviations N of the geoid from the mean sphere of radius R. The two surfaces r_t and r_g intersect each other: their intersections define coastlines that split the Earth surface into dry land Θ_d : $r_t > r_g$ and oceans Θ_w : $r_t < r_g$.

3. SPHERICAL HARMONIC SERIES OF THE TOPOGRAPHICAL POTENTIAL

The topographical masses are generally defined as solid Earth's masses that fill up the space between the geoid and physical surface of the Earth in the dry land domain Θ_d . They are described relatively well geometrically being bounded by the geoid and topographical surface. Since the density distribution of global topographical masses is unknown, the mean value $\rho_a^r = 2,670 \text{ kg} \cdot \text{m}^{-3}$ is usually adopted.

The gravitational potential of the homogeneous topographical masses is given by the volume integral (Martinec and Vaníček, 1994)

$$V^{t}(r,\Omega) = G\rho_{o}^{t} \int_{\Theta} \int_{\xi=r_{g}(\Omega')}^{r_{t}(\Omega)} L^{-1}(r,\Omega,\xi,\Omega') \xi^{2} d\xi d\Omega'$$
(6)

Obviously, it is expected that $r_t \ge r_g$ over dry land Θ_d and $r_t = r_g$ over sea areas Θ_w . Substituting Eq. (2) into (6) and assuming the mean topographical mass density ρ_a^r , the topographical potential reads

$$V^{t}(r,\Omega) = G\rho_{o}^{t} \sum_{n,m} \left(\frac{1}{r}\right)^{n+1} \frac{1}{2n+1} Y_{n,m}(\Omega) \cdot (7)$$
$$\cdot \int_{\Theta} Y_{n,m}^{*}(\Omega') \int_{\xi=r_{g}(\Omega)}^{r_{t}(\Omega)} \xi^{n+2} d\xi d\Omega'.$$

The summation and integration can be interchanged as long as the series is uniformly convergent. Since the computation point is assumed to be outside the gravitating masses, there is no problem with satisfying this requirement. The radial integral in Eq. (7) can be evaluated, cf. (Tsoulis, 2001),

$$\int_{\xi=R+N(\Omega)}^{R+H(\Omega)} \xi^{n+2} d\xi = R^{n+3} F'(\Omega) , \qquad (8)$$

with the following approximation

$$F'(\Omega) \approx \frac{H(\Omega) - N(\Omega)}{R} + (n+2) \frac{H^2(\Omega) - N^2(\Omega)}{2R^2} + (n+2)(n+1) \frac{H^3(\Omega) - N^3(\Omega)}{6R^3}.$$
(9)

The function F' in Eq. (9) describes the geometry of the topographical masses. Higher-order terms were neglected in the binomial series of Eq. (8). Its convergence is discussed in Section 6. The function F'can also be written as the spherical harmonic series

$$F^{t}\left(\Omega\right) = \sum_{n,m} F^{t}_{n,m} Y_{n,m}\left(\Omega\right), \qquad (10)$$

where the spherical harmonic coefficients read

$$F_{n,m}^{t} \approx \frac{H_{n,m} - N_{n,m}}{R} + (n+2) \frac{H_{n,m}^{(2)} - N_{n,m}^{(2)}}{2R^{2}} + (n+2)(n+1) \frac{H_{n,m}^{(3)} - N_{n,m}^{(3)}}{6R^{3}}.$$
(11)

Numerical coefficients $H^{(2)}_{n,m}$ and $H^{(3)}_{n,m}$ are obtained by global spherical harmonic analysis of the height function H squared and cubed. Similarly, spherical harmonic coefficients $N^{(2)}_{n,m}$ and $N^{(3)}_{n,m}$ are evaluated by synthesizing N, computing its powers followed by their analysis. Software tools for spherical harmonic analysis and synthesis up to degree and order 10,800 (respective equiangular resolution of 1 arcmin) are readily available.

Limiting the series by maximum degree nax = 2159, the potential in Eq. (6) can be converted into

$$V^{t}(r,\Omega) \approx GR^{2} \rho_{o}^{t} \sum_{n,m}^{\max} \left(\frac{R}{r}\right)^{n+1} \frac{1}{2n+1} Y_{n,m}(\Omega) \cdot \int_{\Theta} F^{t}(\Omega') Y_{n,m}^{*}(\Omega') d\Omega' , \qquad (12)$$

that can further be developed as follows:

$$V^{t}(r,\Omega) = GR^{2}\rho_{o}^{t}\sum_{n,m}^{nax} \left(\frac{R}{r}\right)^{n+1}\frac{1}{2n+1}Y_{n,m}(\Omega) \cdot \sum_{n'm'}^{nax}F_{n',m'}^{t}\int_{\Theta}Y_{n',m'}(\Omega')Y_{n,m}^{*}(\Omega')d\Omega'$$
(13)

The well known orthogonality of the fullynormalized spherical harmonic functions then yields the topographical potential in the form

$$V^{t}(r,\Omega) = = 4\pi G R^{2} \rho_{o}^{t} \sum_{n,m}^{nax} \left(\frac{R}{r}\right)^{n+1} \frac{1}{2n+1} F_{n,m}^{t} Y_{n,m}(\Omega) .$$
⁽¹⁴⁾

In order to link the spherical harmonic coefficients of the topographical potential with those of the geopotential in EGM08, it would be desirable to express the topographical potential using the same scale factor, namely the geocentric gravitational constant *GM*. Let us assume for simplicity the same

radius of the mean sphere *R*. The geocentric gravitational constant reads for the spherical Earth with the homogeneous mass density $\rho^e = 5,500 \text{ kg} \cdot \text{m}^{-3}$

$$GM = \frac{4}{3} GR^3 \pi \rho^e . \tag{15}$$

The topographical potential finally takes the form

$$V^{t}(r,\Omega) = \frac{GM}{R} \sum_{n,m}^{nax} \left(\frac{R}{r}\right)^{n+1} V^{t}_{n,m} Y_{n,m}(\Omega) , \quad (16)$$

with the spherical harmonic coefficients

$$V_{n,m}^{t} = \frac{3}{2n+1} \frac{\rho_{o}^{t}}{\rho^{e}} F_{n,m}^{t} .$$
 (17)

4. SPHERICAL HARMONIC SERIES OF THE SEA WATER POTENTIAL

Sea water is filling up the space between the ocean bottom r_t and the geoid over the domain Θ_w . Its corresponding gravitational potential can be written as

$$V^{s}(r,\Omega) =$$

$$= G \int_{\Theta} \int_{\xi=r_{t}(\Omega')}^{r_{g}(\Omega')} \rho^{s}(\xi,\Omega') L^{-1}(r,\Omega,\xi,\Omega') \xi^{2} d\xi d\Omega'$$
(18)

In this case, $r_t = r_g$ over dry land Θ_d and $r_t \le r_g$ over sea areas Θ_w . The mass density ρ^s of ocean water at the sea surface is about 1.025 kg·m⁻³. The density continuously increases with decreasing temperature and increasing salinity of sea water with temperature having a greater effect on the density of sea water than salinity does. The deep ocean is layered with the densest water on bottom and the lightest water on top. Sea water tends to move horizontally along isopycnals (surfaces of equal density). The density of ocean water is rarely measured directly but it can be derived from an equation called the equation of state of sea water if salinity, temperature and pressure have been measured. The mass density of ocean water increases almost linearly to the depth of about 1 km where the density equals to 1,028 kg·m⁻³. For larger depths, the mass density of ocean water does not change significantly and can be considered as constant.

The following simple model for the sea water density can be adopted for depths up to 1 km

$$\rho^{s}(\xi,\Omega) \approx \rho^{s}(\xi) = \rho^{s}_{o} \left[1 + \alpha \xi\right] = \rho^{s}_{o} f^{s}(\xi) , \quad (19)$$

with $\rho_o^s = 1.025 \text{ kg m}^{-3}$. The value of the parameter α (m⁻¹) computed as an average linear gradient of the sea water density within the 1 km depth is -2.9268 × 10⁻⁶ m⁻¹. The constant density model of 1,028 kg·m⁻³ can be then used for depths larger than 1 km.

In the following the sea water potential for the upper 1 km layer of sea water is only considered. The remaining part (sea water below 1 km) can be computed by using the formulas from the previous section derived for masses of constant density. For $r_t \ge (r_g - l \text{ km})$ the sea water potential is given by the integral

$$V^{s}(r,\Omega) =$$

$$= G\rho_{o}^{s} \int_{\Theta} \int_{\xi=r_{t}(\Omega')}^{r_{g}(\Omega')} f^{s}(\xi) L^{-1}(r,\Omega,\xi,\Omega') \xi^{2} d\xi d\Omega' .$$
(20)

Substituting for L^{-1} from Eq. (2), one has to solve the inner radial integral

$$\int_{\xi=R+H(\Omega)}^{R+N(\Omega)} f^{s}(\xi) \xi^{n+2} d\xi = R^{n+3} F^{s}(\Omega) .$$
 (21)

Using the binomial expressions expanded into the series truncated at the cubic term, the spherical harmonic coefficients of the function F^{ss} can be derived by following the approach from the previous section, see Eqs. (9)-(11). Substituting for f^s and splitting the function under the integral yields

$$F_{n,m}^{s} \approx \frac{N_{n,m} - H_{n,m}}{R} + (n + 2 - \alpha R) \frac{N_{n,m}^{(2)} - H_{n,m}^{(2)}}{2R^{2}} + \left[(n + 2)(n + 1) - 2(n + 2)\alpha R \right] \frac{N_{n,m}^{(3)} - H_{n,m}^{(3)}}{6R^{3}} .$$
(22)

Limiting the series by maximum degree nax = 2159 results in the series for the sea water potential

$$V^{s}(r,\Omega) = GR^{2}\rho_{o}^{s}\sum_{n,m}^{nax}\left(\frac{R}{r}\right)^{n+1}\frac{4\pi}{2n+1}F_{n,m}^{s}Y_{n,m}(\Omega).$$
(23)

Finally, the sea water potential becomes

$$V^{s}(r,\Omega) = \frac{GM}{R} \sum_{n,m}^{nax} \left(\frac{R}{r}\right)^{n+1} V^{s}_{n,m} Y_{n,m}(\Omega) , \qquad (24)$$

with the spherical harmonic coefficients

$$V_{n,m}^{s} = \frac{3}{2n+1} \frac{\rho_{o}^{s}}{\rho^{e}} F_{n,m}^{s} .$$
 (25)

5. SELECTED FUNCTIONALS OF THE GRAVITATIONAL POTENTIAL

Having defined the harmonic series expansions for the gravitational potentials of the topographical masses and ocean water and having derived their corresponding numerical coefficients, functionals of these potentials can be computed as well. In this contribution, we will focus on selected directional derivatives of the potentials, namely the first- (D_r) and second-order (D_r^2) radial derivatives. For the general potential, e.g., in Eq. (3), we can define these functionals as follows:

$$D_r V(r, \Omega) = -\frac{GM}{R^2} \sum_{n,m}^{nax} (n+1) \left(\frac{R}{r}\right)^{n+2} V_{n,m} Y_{n,m}(\Omega) , \qquad (26)$$

and

$$D_{r}^{2}V(r,\Omega) = \frac{GM}{R^{3}} \sum_{n,m}^{nax} (n+1)(n+2) \left(\frac{R}{r}\right)^{n+3} V_{n,m} Y_{n,m}(\Omega)$$
(27)

Comparing the series for the potential with those for its radial derivatives, it is obvious that the magnitude of their degree variances is largely affected by the eigenvalue (n + 1)/R for the first-order radial derivative and $(n + 1) (n + 2)/R^2$ for the second-order radial derivative. The convergence of the series thus depends on the distance of the computation point from the geocenter (radius *r*), degree *n* as well as the power of the attenuation factor R/r, respectively. This issue is further discussed in Section 6.

Values of the first-order radial derivatives can be used for reduction of gravity observations. In fact, the radial derivative is only an approximation of the topographical correction since observed gravity does not represent the radial derivative of the geopotential. Values of the second-order radial derivatives could then be applied for reduction of spaceborne gradiometric data measured by GOCE.

6. COMPUTATIONAL ASPECTS AND RESULTS

In Sections 3 and 4 the harmonic series representations of the gravitational potentials were formulated. Moreover, the first- and second-order radial derivatives of these harmonic expansions were defined in Section 5 as representative functionals of the gravitational potentials applied routinely in geodesy and geophysics. In the following these series are investigated with respect to their numerical applicability, namely in terms of convergence/divergence with the increasing degree of the harmonic expansion n and increasing geocentric radius r.

Already in some early investigations, the binomial in Eq. (8) was analyzed by, e.g., (Rummel at al., 1988; Vaníček et al., 1995). For low-degree spherical series (such as *n*=180 in case of Rummel et al., 1988), no problems were reported in this regard. However, this situation has changed rather dramatically in recent years since recent global topographical models allow for derivation of corresponding harmonic models up to degree and order of approximately 10,000 (respective angular resolution at the level of 1 arcmin). In this article, the 2 arcmin global relief model of the Earth's surface ETOPO2 is considered. The series to be investigated can be summarized as follows:

$$\sum_{n=0}^{nax} \frac{1}{(2n+1)(n+3)} \left(\frac{R}{r}\right)^{n+1} \sum_{m=0}^{n} \sum_{k=1}^{\infty} \binom{n+3}{k} \frac{H_{n,m}^{(k)}}{R^{k}},$$
(28)

$$\sum_{n=0}^{nax} \frac{n+1}{(2n+1)(n+3)} \left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^{n} \sum_{k=1}^{\infty} \binom{n+3}{k} \frac{H_{n,m}^{(k)}}{R^k} ,$$
(29)

$$\sum_{n=0}^{nax} \frac{(n+1)(n+2)}{(2n+1)(n+3)} \left(\frac{R}{r}\right)^{n+3} \sum_{m=0}^{n} \sum_{k=1}^{\infty} \binom{n+3}{k} \frac{H_{n,m}^{(k)}}{R^k} .$$
(30)

The series in Eq. (28) applies to the gravitational potentials in Eqs. (16) and (24), the other two series in Eqs. (29) and (30) refer then to their first- and second-order radial derivatives, respectively. In the series, some approximations were applied, namely the spherical approximation of the geoid was used and sea water density was approximated by its global mean value. Two different geocentric radii of the computation level were used, namely r = R and r = R + 250 km.

In Figures 1-3, the power spectra (signal degree variances) for terms in the series replacing the binomial expressions in the topographical potential up to the 5th order (i.e., k = 1-5) and its first- and second-order radial derivatives for r = R are shown. Figures 4-6 depict the very same quantities for r = R + 250 km. These figures demonstrate the behavior of the series expansions based on the binomial expressions. Obviously they have problems with convergency when a computation point is located at the sphere of radius R. In this case the numerical study indicates that the truncated series can be used for $n \approx 2000$ (i.e., for angular resolutions up to the level of approximately 5 arcmin) without any problem. The problem vanishes with an increasing distance from the geocenter: for typical low orbiting satellites (such as ESA's GOCE) the series replacing the binomial expression can safely be truncated. Moreover, the power of all terms decreases very fast and for degrees of the spherical harmonic expansion relevant for spaceborne data, the spherical harmonic series can also safely be truncated.

Numerical results related to Sections 4 and 5 are discussed in this paragraph. We start with the topographical potential derived in Section 4. Its discrete numerical values synthesized at the global 5 arcmin equiangular coordinate grid are shown in Figure 7 using the Cartesian linear projection (Wessel and Smith, 1991). Numerical values of its first- and second-order radial derivatives evaluated at the same coordinate grid can be found in Figures 8 and 9, respectively. Figure 10 then represents synthesized numerical values of the sea water potential given again at the global 5 arcmin equiangular coordinate grid. Its first- and second-order radial derivatives can be found in Figures 11 and 12, respectively. Values of the potential and its first-order radial derivative refer to the Earth's surface approximated by the geocentric sphere of radius R. Values of the second-order derivative refer to the radius R + 250 km. The difference in their resolution caused by altitudedependent attenuation is clearly visible.

Both gravitational potentials nicely correspond to the respective mass distributions, i.e., major mountain ranges as well as deep oceanic trenches and other features can nicely be traced in the plotted functions. Numerical values of the first-order radial derivatives reflect changes of the potential functions in the radial direction (gravitation in the radial direction). Both radial derivatives are more complex than their respective potentials, still correlations with height and depth can clearly be recognized. For the topographical masses the loss of a detail in Figure 8 over areas covered by ice sheets (Greenland, Antarctica) is caused by the elevation model used for derivation of the potential coefficients (ETOPO2). Numerical values of the second-order radial derivatives of the gravitational potentials are compatible with some previous studies such as (Wild and Heck, 2008).

7. CONCLUSIONS

The gravitational potentials of solid topography and sea water were formulated in terms of spherical harmonic expansions. Their corresponding spherical harmonic coefficients up to degree and order 5400 were derived by the spherical harmonic analysis of the global relief model ETOPO2. Based on these formulations their first- and second-order radial derivatives can easily be derived and computed as well. The spherical harmonic series rely on the truncated expansion of binomial expressions that must be investigated for convergency. Numerical studies revealed that the series diverged when the values were evaluated at the surface of the sphere of radius R. For the radial derivatives the problem can already be encountered for lower degrees. For typical elevation of low orbiting satellites the harmonic series can safely be truncated (bellow degree 1000), i.e., there is no problem regarding diverging series.

The two gravitational potentials and their firstand second-order radial derivatives were evaluated numerically deploying spherical harmonic coefficients up to degree and order 2159 (5 arcmin angular resolution). Numerical values of all functions computed at the global 5 arcmin equiangular coordinate grid (9,331,200 values) illustrate quite nicely the spatial behavior of the functions. The method and computed values demonstrate the capability of the harmonic series in forward modeling of gravitational fields based on geometric and mass density description of gravitating masses. The methodology is fully compatible with the global gravitational models given by a set of spherical harmonic coefficients of the geopotential (such as the latest EGM08). The spherical harmonic models of the topographical and sea water gravitational fields represent constituents of such global models that can be used for their reduction or stripping. Other mass constituents (ice mass, lake water, ocean sediments, upper layers of the mantle etc.) can be modeled as well. However, one has to be careful about different approximations that lead to some limitations of this method.

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REFERENCES

- Abramowitz, M. and Stegun, I.A.: 1972, Handbook of mathematical functions with formulas, graphs, and mathematical tables. Dover, 9th printing.
- Anderson, E.G.: 1976, The effect of topography on solutions of Stokes's problem. UNISURV Rep. S14, University of New South Wales, Kensington, Australia.
- ETOPO2v2 Global Gridded 2-minute Database, National Geophysical Data Center, National Oceanic and Atmospheric Administration, U.S. Dept. of Commerce.
- Heiskanen, W.A. and Moritz, H.: 1967, Physical geodesy. Freeman and Co., San Francisco.
- MacMillan, W.D.: 1958, Theory of the potential. Dover Publications, New York.
- Mader, K.: 1951, Das Newtonsche Raumpotential prismatischer Körper und seine Ableitungen bis zur dritten Ordnung. Österreichische Zeitschrift für Vermessungswesen, Sonderheft, 11.
- Martinec, Z. and Vaníček, P.: 1994, The indirect effect of topography in the Stokes-Helmert technique for a spherical approximation of the geoid. Manuscripta Geodaetica, 19, 213–219.
- Pavlis, N.K., Holmes, S.A., Kenyon, S.C. and Factor, J.K.: 2008, An Earth Gravitational Model to degree 2160: EGM2008. Presented at the 2008 General Assembly of the European Geosciences Union, Vienna, Austria, April 13-18, 2008.
- Rummel, R., Rapp, R.H., Sünkel, H. and Tscherning, C.C.: 1988, Comparison of global topographic/isostatic models to the Earth's observed gravitational field. Report No. 388, The Ohio State University, Columbus, Ohio, 43210–1247.
- Torge, W.: 2001, Geodesy. Walter de Gruyter, Berlin New York, 3rd edition.
- Tsoulis, D.: 2001, A comparison between the Airy/Heiskanen and the Pratt/Hayford isostatic models for the computation of potential harmonic coefficients. Journal of Geodesy, 74, 637–643.
- Vaníček, P., Najafi, M., Martinec, Z., Harrie, L. and Sjöberg, L.E.: 1995, Higher-degree reference field in the generalized Stokes-Helmert's scheme for geoid computation. Journal of Geodesy, 70, 176–182.
- Wessel, P. and Smith, W.H.F.: 1991, Free software helps map and display data. EOS Trans. AGU 72, 441.
- Wild, F. and Heck, B.: 2008, Topographic and isostatic reductions for use in satellite gravity gradiometry. International Association of Geodesy Symposia, 132, 49–55.



Fig. 1 Potential – series power spectrum for H = 0 km.

Fig. 2 First-order radial derivative – series power spectrum for H = 0 km.



Fig. 3 Second-order radial derivative – series power spectrum for H = 0 km.

Fig. 4 Potential – series power spectrum for H = 250 km.



Fig. 5 First-order radial derivative – series power Fig. 6 spectrum for H = 250 km.

6 Second-order radial derivative – series power spectrum for H = 250 km.







Fig. 8 First-order radial derivative of the topographical potential at $H = 0 \text{ km} (\text{mGal} = 10^{-5} \text{ m} \cdot \text{s}^{-2})$



Fig. 9 Second-order radial derivative of the topographical potential at H = 250 km (E = 10^{-9} s⁻²).



Fig. 10 Sea water potential at $H = 0 \text{ km} (\text{GPU} = \text{m}^2 \cdot \text{s}^{-2})$.



Fig. 11 First-order radial derivative of the sea water potential at $H = 0 \text{ km} (\text{mGal}=10^{-5} \text{ m} \cdot \text{s}^{-2})$.



Fig. 12 Second-order radial derivative of the sea water potential at H = 250 km (E=10⁻⁹ s⁻²).