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LOCAL QUASIGEOID MODELLING USING GRAVITY DATA INVERSION TECHNIQUE - ANALYSIS OF FIXED COEFFICIENTS OF DENSITY MODEL WEIGHTING MATRIX

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ABSTRACT
The paper presents analysis relating to the method of local quasigeoid modelling based on the geophysical technique of gravity data inversion, using non-reduced surface gravity data and GNSS/levelling height anomalies. One of the main problems occurring in the application of the method is to determine the model weighting matrix, the purpose of which is to control the inversion process. This paper presents the analyses concerning the determination of certain constant coefficients signed as α_Ω , α_κ and β, appearing in the definition of the model weighting matrix. The calculations performed indicate that because of the accuracy of the density model, the coefficient β should be in the range of $0.001 \le \beta \le 0.01$, and the range $0.0025 \le \beta \le 0.005$ should be adopted as the optimal. As the optimal values of the coefficients α_Ω and α_κ, values $\alpha_\kappa = 0.1$ and $\alpha_\Omega = 0.01$, for the zones of constant density with area less than about 130 km^2 were determined.

KEYWORDS: local quasigeoid model, topographical mass density distribution model

1. INTRODUCTION

The paper Trojanowicz (2007) proposed an approach to local modelling of the quasigeoid, based on the solution of the linear problem of gravity data inversion. The method has a potential to use unreduced gravity data in the form of gravity disturbances or free air anomalies defined on the earth's surface and measured GNSS/levelling height anomalies. Test results of sample calculations, presented in the papers Trojanowicz (2007; 2012), pointed to several advantages of this approach. First, should be mentioned a local modelling of the quasigeoid using local data and lower requirements with respect to the amount of needed gravity data and resolution of a digital terrain model in comparison to the established, integral methods. It should be emphasized that in these calculations very high accuracy of the developed models was obtained (at the level of precision of GNSS/levelling test data). However, the applied method of calculation does not have a final, closed-form. There are several elements that require deeper analysis. This study represents another step towards a final, stable solution and concerns the detailed analysis of certain elements of the algorithm. Although the algorithm was described in details in the papers Trojanowicz (2007; 2012), due to the need to highlight the problem covered by this study, it will be recalled again.

2. DESCRIPTION OF THE ANALYZED APPROACH

The analyzed method is based on a certain, local model of disturbing potential on the earth's surface and in outer space. The required input data are the previously mentioned gravity anomalies or gravity disturbances and the disturbing potential, calculated on the basis of GNSS/levelling height anomalies. The height anomalies are converted into the disturbing potential (as well as the reverse conversion) based on the Bruns formula (see e.g. Torge, 2001).

Following Trojanowicz (2007; 2012), let's consider a point P situated on a terrain surface (Fig. 1). Disturbing potential at that point can be split into three components:

- potential T_Ω produced by topographic masses Ω lying above the geoid, with density distribution function ρ.
- potential T_κ produced by disturbing masses κ occurring under the geoid surface (down to the compensation level) with density distribution function δ.
- potential T_r which represents the remaining influences.

Potentials T_Ω and T_κ are calculated by Newton's integral and expressed respectively:

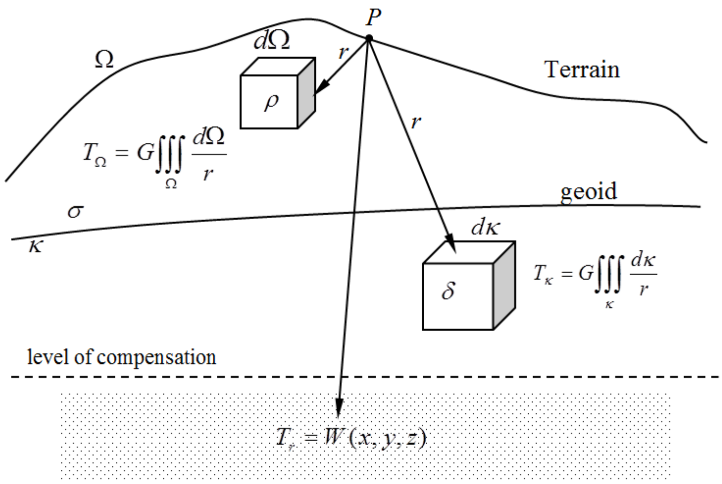


Fig. 1 Disturbing potential model.

$$T_\Omega = G\iiint_\Omega \frac{\rho}{r} dV_\Omega \tag{1}$$

$$T_\kappa = G\iiint_\kappa \frac{\delta}{r} dV_\kappa \tag{2}$$

where G is the Newton's gravitational constant, dV_Ω and dV_κ are elements of volume, and r is the distance between the attracting masses and the attracted point P.

Regions Ω and κ cover a limited area of interest. Because the data used for calculations (gravity anomalies and disturbances, and GNSS/levelling height anomalies) contain information about density distribution also outside these regions (including masses lying below the compensation level), the potential T_r is introduced into the calculations. The role of the potential T_r is to model this "unwanted" part of the data. Furthermore in practice, each model of the gravimetric geoid or quasigeoid (based only on gravity data) indicates a systematic offset and tilt with reference to GNSS/levelling data, caused by the long-wavelength errors of geoid or quasigeoid model and systematic errors of both levelling and satellite data (Tscherning, 2001). So, another role of the potential T_r is to link the gravity and the GNSS/levelling data - it has to cover the previously mentioned offset and tilt between the gravimetric quasigeoid and GNSS/levelling data. It was assumed for calculations that both the distorting effects mentioned can be modeled by harmonic polynomials of a low degree.

The disturbing potential on the terrain surface finally, can be written as:

$$T_P = T_r + T_\Omega + T_\kappa \tag{3}$$

Based on Eq. 3, inversion task is formulated in the form: finding the density distribution functions ρ and δ in defined areas Ω and κ, and deriving the coefficients of the polynomials modelling the potential T_r to satisfy equation (3).

Solution of the given task using linear inversion, following Blakely (1995), requires discretization of the continuous 3D functions ρ and δ. The regions Ω and κ are therefore divided into finite volume blocks and a constant density (which become the investigated value) is then assigned to each of the blocks.

In studies completed until now, it is assumed that the division of area Ω is defined by a digital terrain model (DTM) in the form of rectangular blocks. Since the determination of the density for each block of the DTM would require calculations of a large number of unknowns, DTM blocks are grouped into zones of constant density. Thus the investigated, zone density ρ_k refers to all of the DTM blocks situated in zone k.

The area κ is treated in calculations as a plate of constant thickness. This plate is divided into blocks of constant density δ_j, which form a set of one or more layers of cuboids.

All calculations are performed in a local Cartesian coordinate system. The Z-axis of the coordinate system is directed towards the geodetic Zenith and the X and Y axes lay on the plane of the horizon and are directed to the North and East, respectively. The origin of the coordinate system can be set in the middle of the area. In this case the Ω and κ regions are defined as a rectangular grid of rectangular prisms (Fig. 2).

After defining the areas Ω and κ, the potential (3) can be written as a linear function of unknown parameters: ρ_k and δ_j, and the coefficients of

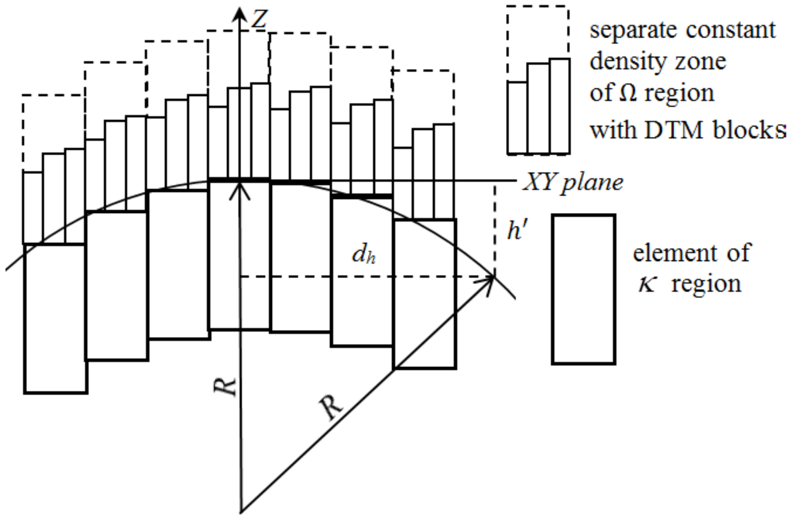


Fig. 2 Ω and κ regions in the Cartesian coordinate system.

polynomials describing the potential T_r. Assuming a fixed form of the polynomials, the potential (3) can be written in the form:

$$T_P = T_r + T_\Omega + T_\kappa = \mathbf{w}^\mathbf{T}\mathbf{a} + \mathbf{t}^\mathbf{T}\boldsymbol{\rho} + \mathbf{r}^\mathbf{T}\boldsymbol{\delta} \tag{4}$$

where:

$$T_r = a_1 + a_2 X_P + a_3 Y_P + a_4 X_P Y_P + a_5 Z_P = \mathbf{w}^\mathbf{T}\mathbf{a} \text{ [1]} \tag{5}$$

X_P, Y_P, Z_P - coordinates of point P

$\mathbf{a}^\mathbf{T} = [a_1, ..., a_l]$ is the vector of polynomials coefficients a_u ($u=1...l$),

$\mathbf{w}^\mathbf{T} = [w_1, ..., w_l]$ depends on used polynomial,

$$\mathbf{t}^\mathbf{T}\boldsymbol{\rho} = T_\Omega(P) = \sum_{k=1}^{n}\left(\rho_k G \sum_{i=1}^{m_k} \int_{z_{i1}}^{z_{i2}} \int_{y_{i1}}^{y_{i2}} \int_{x_{i1}}^{x_{i2}} \frac{1}{d_i} dx_i dy_i dz_i\right) \tag{6}$$

$$d_i = \sqrt{(x_i - X_P)^2 + (y_i - Y_P)^2 + (z_i - Z_P)^2},$$

n – number of DTM zones (where for each zone the constant density ρ_k will be calculated), m_k – number of rectangular prisms of the DTM in zone k, $x_{i1}, x_{i2}, y_{i1}, y_{i2}, z_{i1}, z_{i2}$ - coordinates defining rectangular prism i of DTM, $\boldsymbol{\rho}^\mathbf{T} = [\rho_1, ..., \rho_n]$ is the vector of constant densities of topographic masses, $\mathbf{t}^\mathbf{T} = [t_1, ..., t_n]$ comes from (6) and values t_k ($k = 1...n$) are defined as:

$$t_k = G \sum_{i=1}^{m_k}\left(\int_{z_{i1}}^{z_{i2}} \int_{y_{i1}}^{y_{i2}} \int_{x_{i1}}^{x_{i2}} \frac{1}{d_i} dx_i dy_i dz_i\right) \tag{7}$$

$$\mathbf{r}^\mathbf{T}\boldsymbol{\delta} = T_\kappa(P) = \sum_{j=1}^{s}\left(\delta_j G \int_{z_{j1}}^{z_{j2}} \int_{y_{j1}}^{y_{j2}} \int_{x_{j1}}^{x_{j2}} \frac{1}{d_j} dx_j dy_j dz_j\right) \tag{8}$$

$$d_j = \sqrt{(x_j - X_P)^2 + (y_j - Y_P)^2 + (z_j - Z_P)^2},$$

s – number of rectangular prisms of the κ area; $x_{j1}, x_{j2}, y_{j1}, y_{j2}, z_{j1}, z_{j2}$ - coordinates defining rectangular prism j of κ area, $\boldsymbol{\delta}^\mathrm{T} = [\delta_1, ..., \delta_s]$ is the vector of densities of region κ, $\mathbf{r}^\mathrm{T} = [r_1, ..., r_s]$ comes from (8) and values r_j ($j=1...s$) are defined as:

$$r_j = G \int_{z_{j1}}^{z_{j2}} \int_{y_{j1}}^{y_{j2}} \int_{x_{j1}}^{x_{j2}} \frac{1}{d_j} dx_j dy_j dz_j \tag{9}$$

Calculations performed according to equations (6) and (8) should take into account earth's curvature. If we assume a local, spherical model of the geoid (Fig. 2), curvature of the earth can be included by designation of coordinates z, which defines either top or bottom surface of the rectangular prism i or j of regions Ω and κ respectively, for each data point. Using the height H of this surface over the geoid it can be written:

$$z = H - h' = H - (R - \sqrt{R^2 - d_h^2}) \tag{10}$$

where R – mean radius of the Earth, d_h – horizontal distance of i or j block center from the survey point

In the Cartesian coordinate system defined above the origin of the Z-axis is always under the actual survey point, at a depth equal to the height of the point and the direction of the Z axis remains constant.

The solutions of the integrals (7) and (9) and their vertical (z) derivatives, that should be used in practical calculations, can be found in e.g. Forsberg

[1] This polynomials were used in all calculations

and Tscherning (1997), Nagy (1966), Nagy et al. (2001).

Inversion of gravity data is usually accomplished by adopting a certain reference density model. When the determined density model would be written as $\boldsymbol{\tau}^T = \lfloor \boldsymbol{\rho}^T, \boldsymbol{\delta}^T \rfloor = \lfloor \rho_1, ..., \rho_n, \delta_1, ..., \delta_s \rfloor$, after adopting a reference model in the form $\boldsymbol{\tau}_0^T = \lfloor \boldsymbol{\rho}_0^T, \boldsymbol{\delta}_0^T \rfloor = \lfloor \rho_1^0, ..., \rho_n^0, \delta_1^0, ..., \delta_s^0 \rfloor$, the vector of unknowns could be written as $\mathbf{dx}^T = \lfloor \mathbf{a}^T, \mathbf{d\tau}^T \rfloor$, where $\mathbf{d\tau}^T = \boldsymbol{\tau} - \boldsymbol{\tau}_0$ is the sought vector of density differences.

On the basis of the disturbing potential model so defined, and taking into account that the problem will be solved by least squares methods, proper observation equations can be formed. The key data are the disturbing potential values determined from GNSS/levelling height anomalies, gravity disturbances and free air anomalies defined on the earth's surface. The observation equations for these data can be written as:

$$T_P + v_{TP} = \mathbf{f}^{\mathrm{T}}\mathbf{dx} + T_P^0$$

$$\Delta g_P + v_{\Delta gP} = \left(-\mathbf{f}_{\mathbf{z}}^{\mathrm{T}} + \frac{U_{zz}}{\gamma_Q}\mathbf{f}^{\mathrm{T}} \right)_P \mathbf{dx} + \Delta g_P^0 \tag{11}$$

where $\mathbf{f}^{\mathrm{T}} = \lfloor \mathbf{w}^{\mathrm{T}}, \mathbf{t}^{\mathrm{T}}, \mathbf{r}^{\mathrm{T}} \rfloor = [w_1, ..., w_l, t_1, ..., t_n, r_1, ..., r_s]$ is the known vector, $\mathbf{f}_{\mathbf{z}}$ is the z derivative of the vector $\mathbf{f}$, $v_{TP}, v_{\Delta gP}, v_{\delta gP}$ are adjustment errors, γ_Q is a normal gravity acceleration on telluroid and U_{zz} is its vertical gradient, $T_P^0 = \mathbf{f}^{\mathrm{T}}\mathbf{x}_0$,

$\Delta g_P^0 = \left(-\mathbf{f}_{\mathbf{z}}^{\mathrm{T}} + \frac{U_{zz}}{\gamma_Q}\mathbf{f}^{\mathrm{T}} \right)_P \mathbf{x}_0$, $\delta g_P^0 = -\mathbf{f}_{\mathbf{z}}^{\mathrm{T}}\mathbf{x}_0$ are the approximate observation quantities determined on the basis of the vector $\mathbf{x}_0^T = \lfloor \mathbf{a}_0^T, \boldsymbol{\tau}_0^T \rfloor$, where the vector $\mathbf{a}_0$ is l-dimension zero vector .

The formulated equations (11), for a series of observations, may be written as:

$$\mathbf{v} = \mathbf{A}\mathbf{dx} - \mathbf{L} \tag{12}$$

where $\mathbf{v}^T = \lfloor v_{TP}, ..., v_{\Delta gP}, ..., v_{\delta gP}, ... \rfloor$ is the vector of adjustment errors,
$\mathbf{L}^T = \lfloor T_P - T_P^0, ..., \Delta g_P - \Delta g_P^0, ..., \delta g_P - \delta g_P^0, ... \rfloor$ is a known observation vector and $\mathbf{A}$ is the design matrix of known coefficients.

The system of equations (12) is solved by the general condition of the least squares:

$$\mathbf{v}^T\mathbf{P}\mathbf{v} = \min \tag{13}$$

where $\mathbf{P}$ is the given weight matrix.

The problem of non-uniqueness of the gravity inversion is solved by introducing the condition, suggested by Li and Oldenburg (1998):

$$\mathbf{d\tau}^T\mathbf{W}_{\boldsymbol{\tau}}\mathbf{d\tau} = \min \tag{14}$$

where $\mathbf{W}_{\boldsymbol{\tau}}$ is the model weighting matrix defined in Li and Oldenburg (1998).

Condition (14) for all unknowns is written as:

$$\mathbf{dx}^T\mathbf{W}_{\mathbf{x}}\mathbf{dx} = \min \tag{15}$$

where $\mathbf{W}_{\mathbf{x}} = \begin{bmatrix} \mathbf{W}_{\mathbf{a}} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{\boldsymbol{\tau}} \end{bmatrix}$, and $\mathbf{W}_{\mathbf{a}}$ is the zero weighting matrix assigned to the vector of polynomial coefficients (5).

The weighting matrix $\mathbf{W}_{\boldsymbol{\tau}}$, in the analyzed solution, is defined as (Trojanowicz, 2012):

$$\mathbf{W}_{\boldsymbol{\tau}} = \mathbf{W}^{\mathbf{d}} + \mathbf{W}^{\mathbf{c}} \tag{16}$$

where matrix $\mathbf{W}^{\mathbf{d}}$ is a diagonal matrix defined by the depth weighing function, elements of which are defined as:

$$\mathbf{W}_{ii}^d = \begin{cases} \alpha_\Omega \sqrt{w_{\Omega\, k}} & for\ \ \Omega \\ \alpha_\kappa \sqrt{w_{\kappa\, j}} & for\ \ \kappa \end{cases} \tag{17}$$

where α_Ω, α_κ are constant coefficients and $w_{\Omega\, k}$, $w_{\kappa\, j}$ equal to the value of the vertical component of gravitation produced by zone k of constant density of the Ω region or cuboid j of region κ, in the point on the terrain surface, above the center of the constant density zone k or cuboid j;

matrix $\mathbf{W}^{\mathbf{c}}$ defines spatial correlations between zones of constant density and is defined as:

$$\mathbf{W}^{\mathbf{c}} = \sum_{i=1}^{n} \sum_{p=s}^{t} \mathbf{C}^{\mathbf{ip}} \tag{18}$$

where $\mathbf{C}^{\mathbf{ip}}$ is a matrix defining correlation between a couple of zones of constant density (i, p).

In the matrix $\mathbf{C}^{\mathbf{ip}}$ only four elements, that correspond to couple zones (i, p), are not equal to zero and are defined as:

$$C_{ii}^{ip} = w_i w_i, \quad C_{ip}^{ip} = w_i w_p, \quad C_{pi}^{ip} = w_p w_i,$$
$$C_{pp}^{ip} = w_p w_p \tag{19}$$

where $w_i = -w_p = \beta \frac{\Delta x \Delta y}{d_{ip}^2}$, β is a constant coefficient, Δx, Δy are mean distances between adjacent zones of constant density in x and y direction, d_{ip} is the distance between centers of zones i and p.

The least square objective function is now written as:

$$\mathbf{v}^T\mathbf{P}\mathbf{v} + \mathbf{dx}^T\mathbf{W}_\mathbf{x}\mathbf{dx} = \min \tag{20}$$

Solution of equation (12), taking into account the condition (20) has the form:

$$\mathbf{dx} = \left(\mathbf{A}^T\mathbf{P}\mathbf{A} + \mathbf{W}_\mathbf{x}\right)^{-1}\mathbf{A}^T\mathbf{P}\mathbf{L} \tag{21}$$

Matrix $\mathbf{W}_\mathbf{x}$ defined by formulas (15-19) depends on three coefficients α_Ω, α_κ, and β adopted in the calculation as constants. The purpose of the paper is an attempt to determine the optimal values of these coefficients, simultaneously enabling determination of the best quasigeoid model and a model of topographic mass densities. Additionally, this paper is focused on analysis of these coefficients. In particular, the study concern the influence of the size of constant density zones and the impact of global geopotential models on values of these coefficients. In test calculations it was assumed that matrix P is defined on the basis of observation errors. Measurement errors were adopted as the errors of GNSS/levelling height anomalies. To estimate the errors of gravity data the initial modelling procedure was used, in which the weights of gravity data were assumed small, in order not to limit the estimated parameters of the model (e.g. $m_{\delta g} = \pm 20\,\text{mGal}$). Then the standard deviation of the adjustment errors of gravity data was estimated. This value was then taken as gravity data errors in the next, main calculations.

In the presented approach calculations can be performed using also the global geopotential model. This variant of calculations uses the remove-compute-restore technique, where the disturbing potential is presented as a sum of the global component T_{GM} and residual potential $T_{\Delta g}$ ($T = T_{GM} + T_{\Delta g}$). The calculations are carried out in three stages (Trojanowicz, 2012):

1. The global component is removed from the data and the residual data are formed: $T_{\Delta g} = T - T_{GM}$ for GNSS/leveling and $\delta g_{\Delta g} = \delta g - \delta g_{GM}$ or $\Delta g_{\Delta g} = \Delta g - \Delta g_{GM}$ for gravity data.
2. Model of residual disturbing potential is build based on the residual data. At the new points the residual disturbing potential values are determined from this model.
3. At the new points global component is restored: $T = T_{GM} + T_{\Delta g}$

In relation to the above, densities are also decomposed into two components $\rho = \rho_{GM} + \rho_{\Delta g}$ and $\delta = \delta_{GM} + \delta_{\Delta g}$. The components ρ_{GM} and δ_{GM} are estimated based on the data T_{GM} and δg_{GM} or Δg_{GM}. The components $\rho_{\Delta g}$ and $\delta_{\Delta g}$ are determined based on the residual data. For both the mentioned steps of density modelling different reference density models (ρ_0, δ_o), can be adopted.

3. TEST CALCULATIONS

The test calculations were carried out in relation to the area of Lower Silesia. For the calculations 1518 gravimetric points[2] were used, covering an area of about 23,000 km^2 (approximately 1 point per 15 km^2) and 28 POLREF[3] network points, with measured GNSS/levelling height anomalies (ζ_{SL}) of accuracy approx. $\pm 2\,cm$ (Kryński et al., 2005). This accuracy was later determined to be too optimistic and estimated at the level of 3-4 *cm* (Kryński, 2007).

Calculations were done in a local Cartesian coordinate system, the origin of which was in the point $\varphi = 51°6'$, $\lambda = 16°30'$ (Fig. 3). The elements of weighting matrix (17) were calculated using a unit density.

The POLREF network points were divided into two groups. The first group consisted of 14 points, marked with triangles in Figure 3, is treated as a group of data points. The remaining 14 points, marked by circles in Figure 3, is a group of test points. The calculation was carried out in several series for different values of the coefficients. In each calculation run, on the basis of 1518 gravity points and 14 GNSS/levelling data points, a quasigeoid model was built. The model accuracy was afterwards estimated, based on 14 test points. To assess the accuracy of the topographic mass density distribution model, a numerical model of topographic mass density (NMTD) was used, which was built on the basis of a map of rock slab density above sea level (Krolikowski and Polechońska, 2005). The density map was also used to draw a sketch of the density of topographic masses shown in Figure 4.

As mentioned above, the calculations were performed for several different values of the coefficients α_Ω, α_κ, and β. Table 1 contains values of coefficients adopted for test calculations.

All calculations were carried out in three variants, i.e. without any global geopotential model (M0) and with the two global geopotential models: EGM96 (Lemoine et al., 1998) and EGM08 (Pavlis et al., 2008) marked as M96 and M08.

[2] Data referred to the International Gravity Standardization Network 1971 (IGSN71), provided by the Polish Geological Institute

[3] POLish REference Frame - data provided by the Head Office of Geodesy and Cartography of Poland

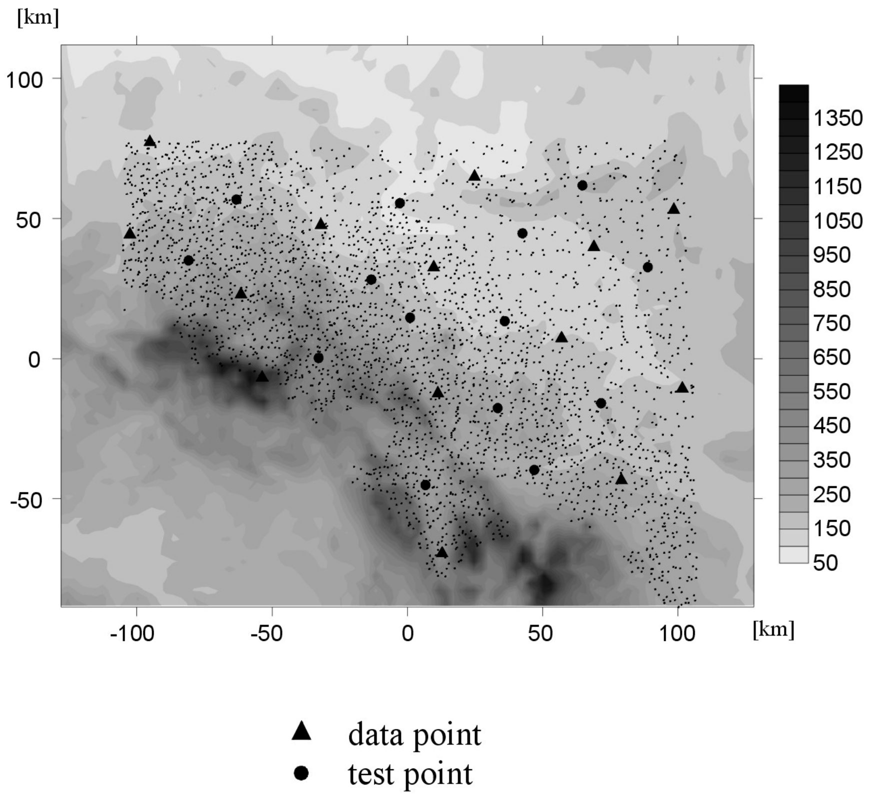


Fig. 3 Test region.

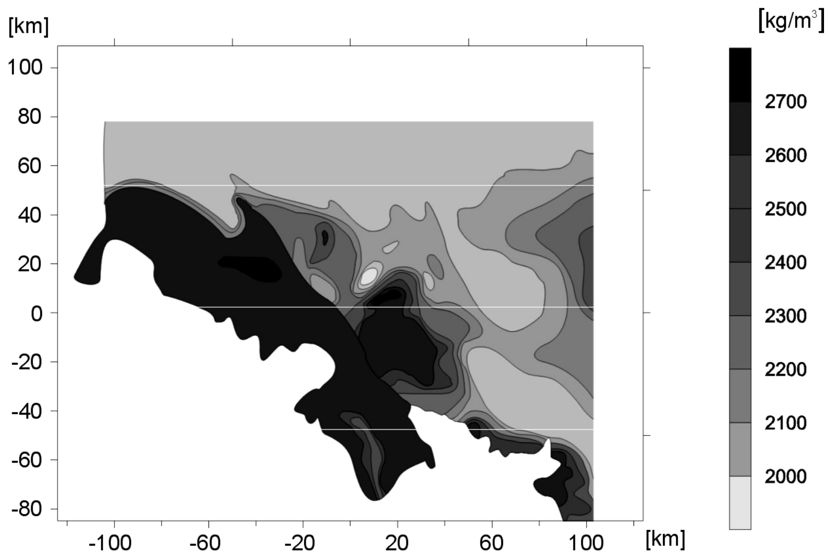


Fig. 4 Theoretical densities of topographic masses (sketch was drawn on the basis of a map of rock slab density above sea level (Królikowski and Polechońska, 2005)).

Table 1 The values of coefficients adopted for test calculations.

α_Ω	100	10	1	0.1	0.01	0.001	0.0001
α_κ	100	10	1	0.1	0.01	0.001	0.0001
β	1	0.1	0.01	0.005	0.0025	0.001	0.0001

Table 2 Size of the analyzed zones of constant density.

Zoning symbol	Single zone dimension [km]	Single zone area [km^2]
Z1	17.0×13.3	227.5
Z2	12.8×10.0	128
Z3	10.2×8.0	81.9
Z4	8.5×6.7	56.9

In variants M0 and M96, for components $\rho_{\Delta g}$ and $\delta_{\Delta g}$, the constant value $\rho_0 = 2200$ kg/m^3 [4] was assumed as the reference density model for the area Ω. Reference density model for the area κ (δ_0) was adopted as "negative density", which balanced topographical masses of the area Ω. So the density of separate prism of the area κ was calculated based on the equation $\delta_j^0 = -\frac{H_i \rho_i^0}{h_j}$, where H_i, ρ_i^0 are mean height and reference density of zone *i* of the Ω area, situated directly above prism *j* of area κ, h_j is height of prism *j*. For modelling the components $\rho_{\Delta g}$ and δ_{GM} the values $\rho_0 = 0$, $\delta_0 = 0$ were adopted as reference density model.

In variant M08, for components $\rho_{\Delta g}$ and $\delta_{\Delta g}$ the values $\rho_0 = 0$ and $\delta_0 = 0$ were adopted as reference density model. For components ρ_{GM} and δ_{GM} constant value $\rho_0 = 2200$ kg/m^3 was assumed for the area Ω and for the area κ, defined above, "negative density" was calculated.

The differences in defining the reference density models were introduced based on a series of preliminary calculations, in which the described above ways of defining the reference density models were mixed. As the final procedure of definition of the reference density model was selected the one which provides the highest accuracy of both topographic mass density and quasigeoid models.

To estimate the correlation between the coefficients and size of constant density zones, the calculations were carried out adopting the dimensions of the zones defined in Table 2.

In order to carry out the analysis for all variants, 4116 computing series were performed. Thus, there was a large number of results obtained. An analysis as well as concise and readable presentation of such a vast amount of data is challenging. Taking this into account, the analysis were carried out in two stages. In the first stage the optimal value of the coefficient β was analyzed and determined. In a second step the optimal values of the coefficients α_Ω and α_κ were estimated.

3.1. ESTIMATION OF OPTIMAL VALUE OF COEFFICIENT β

In order to determine the optimal value of the coefficient β, the calculation results were divided into groups. The main group was a series of 343 calculations performed for all adopted values of coefficients α_Ω, α_κ and β (Table 1), done for one of the assumed zoning (Z1, Z2, Z3 or Z4), and for one variant (M0, M96 or M08). Analyzed groups are therefore 12. For each of the seven values of coefficient β (analyzed in one group), the best result was selected (1 out of 49 determined for different values of the coefficients α_Ω and α_κ). For the analysis two parameters were adopted: error of quasigeoid model m_ζ (calculated based on 14 test points) and error of topographic mass density distribution model m_ρ (calculated on the basis of NMTD).

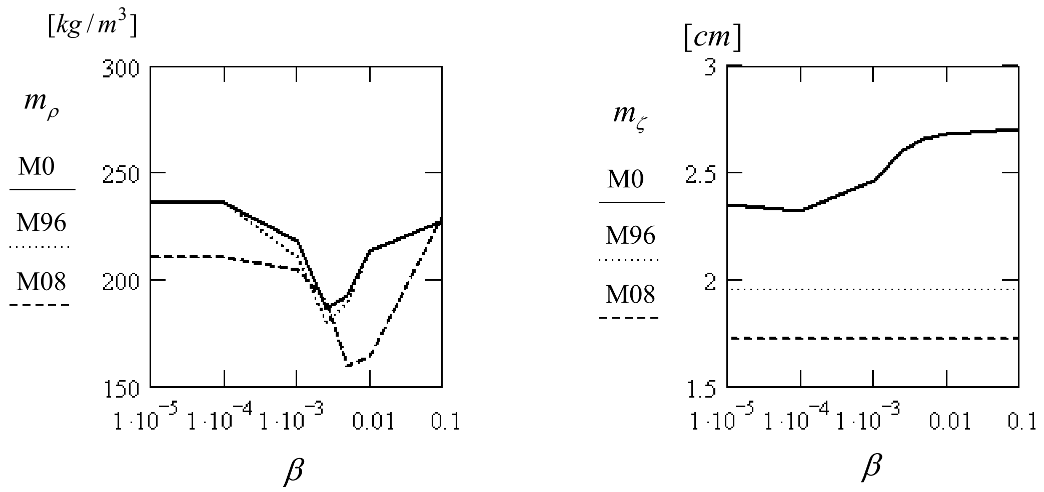
Figure 5 presents graphs of the best parameters m_ρ and m_ζ as a functions of coefficient β.

In Figure 5 the strong influence of coefficient β on accuracy of the density model is presented. Clearly, the best results were obtained in all the variants for $\beta = 0.0025$ or $\beta = 0.005$ and good results for $0.001 \le \beta \le 0.01$.

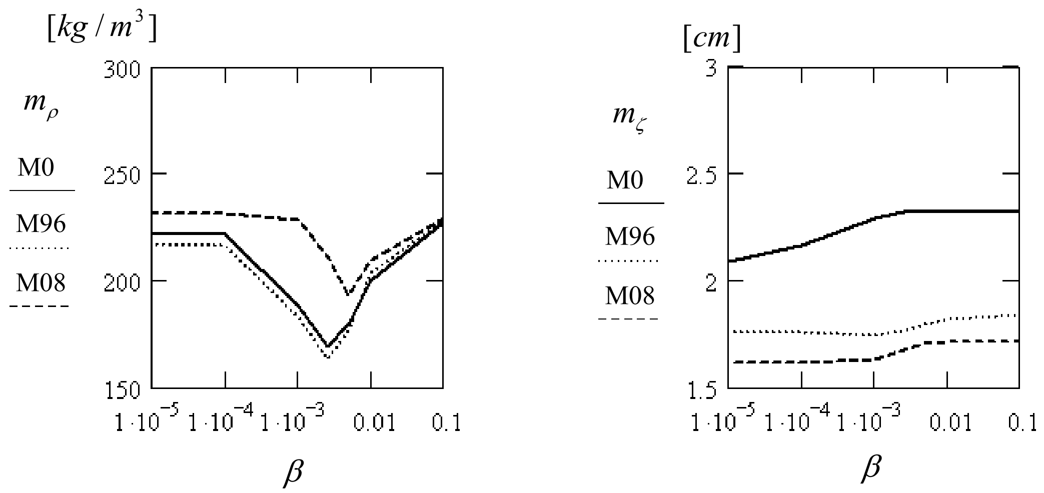
Analyzing the results of calculations concerning the accuracy of quasigeoid model, one can see a very small impact of the coefficient β on the error m_ζ in variants M96 and M08. Considering the optimistic accuracy of GNSS/levelling data of approximately $\pm 2\,cm$, one can observe that for these variants the accuracy of the quasigeoid model does not depend on β (in terms of accuracy of used test data). The M0 variant shows little relation between error m_ζ and coefficient β for the zoning Z1, Z2 and Z3 (a decrease of accuracy for the coefficients $\beta \ge 0.001$ can be observed). The relation is not the case for division of Z4, for which the accuracy of the model is practically the same for all tested values of the β coefficient.

[4] Value close to the mean density of topographic masses for lowland areas of Poland (Królikowski and Polechońska, 2005)

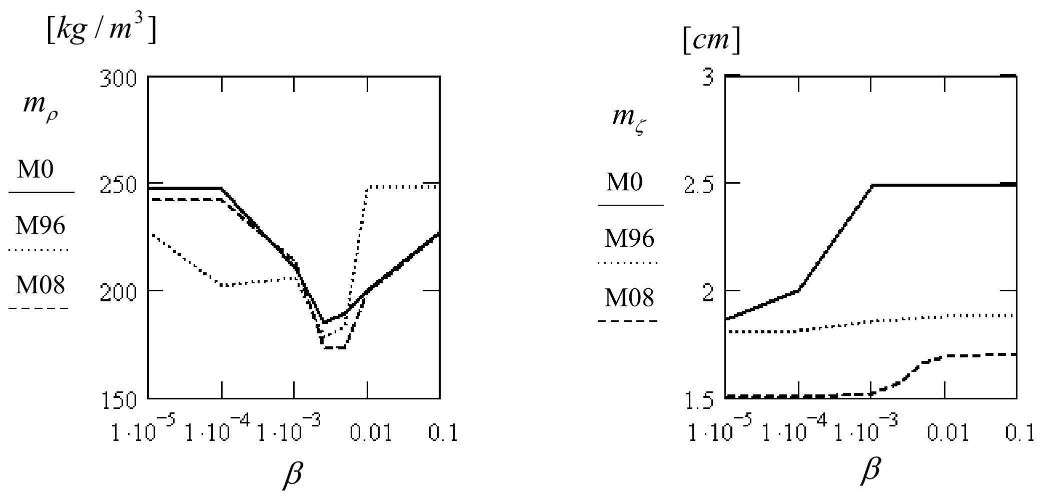
a) zoning Z1



b) zoning Z2



c) zoning Z3



d) zoning Z4

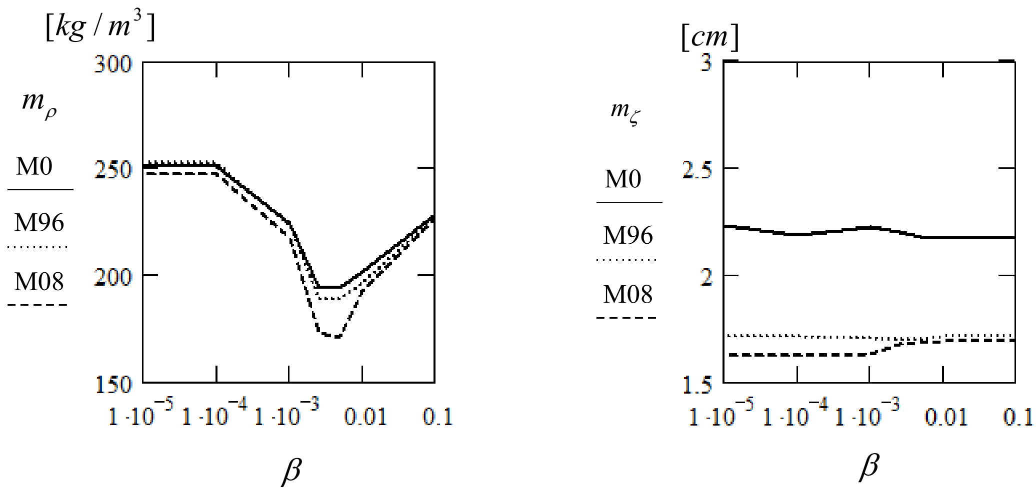


Fig. 5 Graphs of the best parameters m_ρ and m_ζ as a functions of coefficient β .

zoning Z5

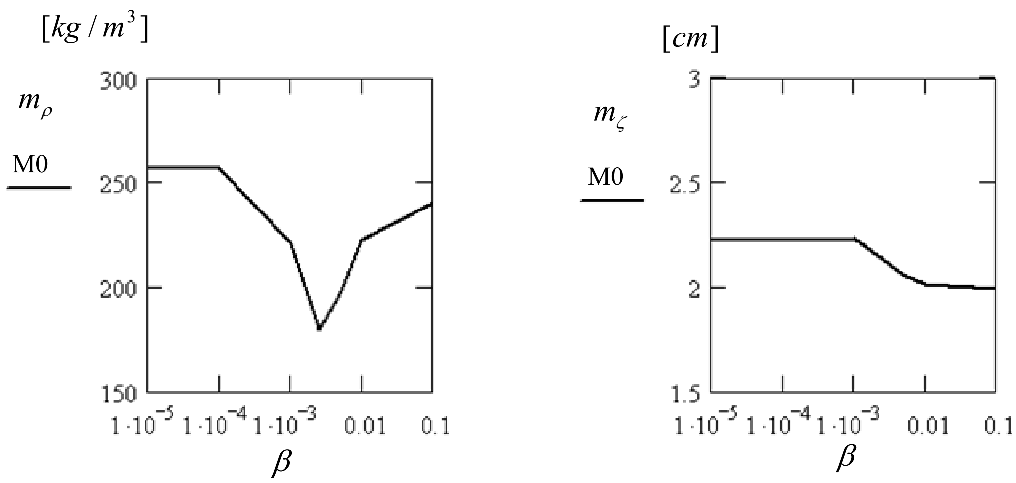


Fig. 6 Graphs of the best parameters m_ρ and m_ζ as a functions of coefficient β , for variant M0, zoning Z5.

In order to clarify the relation between coefficient β and accuracy of quasigeoid model, for variant M0, an additional series of calculations assuming a constant density zone with dimensions of ca. $6.4 \times 5 km$ and area of ca. $32 km^2$ were performed. This zoning is marked as Z5, and similar graphs as in Figure 5 are shown in Figure 6.

Analyzing the graphs it can be stated that dependence of the density model accuracy on coefficient β is similar to the variants previously described. However, the graph of dependence of accuracy of the quasigeoid model on coefficient β is similar to the results obtained for zoning Z4 (omitting the approx. 2 mm increase in accuracy for coefficients $\beta \geq 0.001$).

In conclusion, it should be noted that for the M0 variant the constant density zone size slightly affects the optimal values of coefficient β for larger zones. It can be assumed that for zones of area less than 57 km^2 (zoning Z4) the accuracy of the quasigeoid model does not depend on coefficient β .

Inspecting again Figure 5, far better qasigeoid models should be noted for all variations using global geopotential models. This clearly indicates that in practical calculations the available global geopotential models should definitely be used.

Taking into account the above conclusions, due to the accuracy of the density model, coefficient β should be in the range $0.001 \leq \beta \leq 0.01$. In performed tests the best results were obtained for $\beta = 0.0025$ and $\beta = 0.005$.

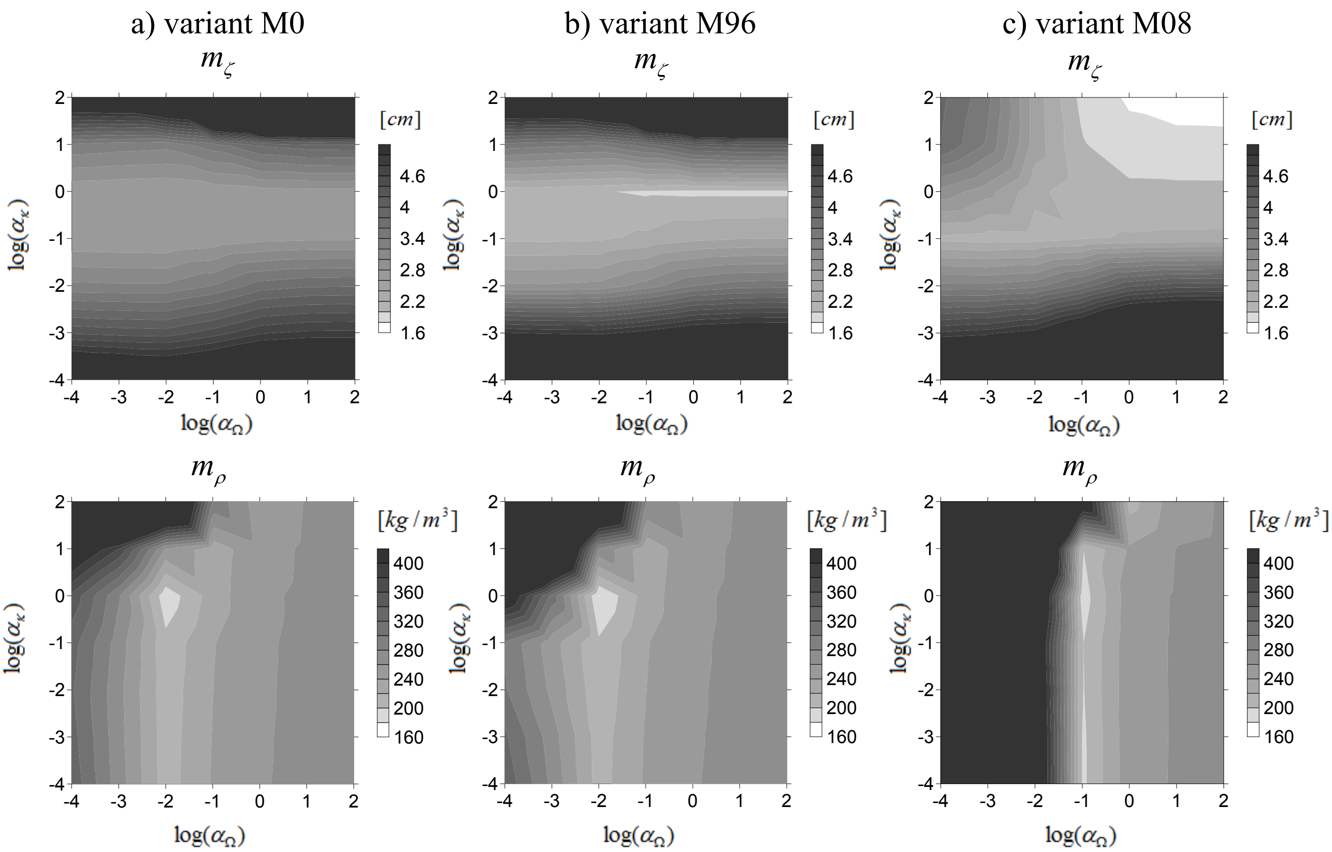


Fig. 7 Parameters m_ζ and m_ρ as a function of coefficients α_Ω and α_κ, for zoning Z1.

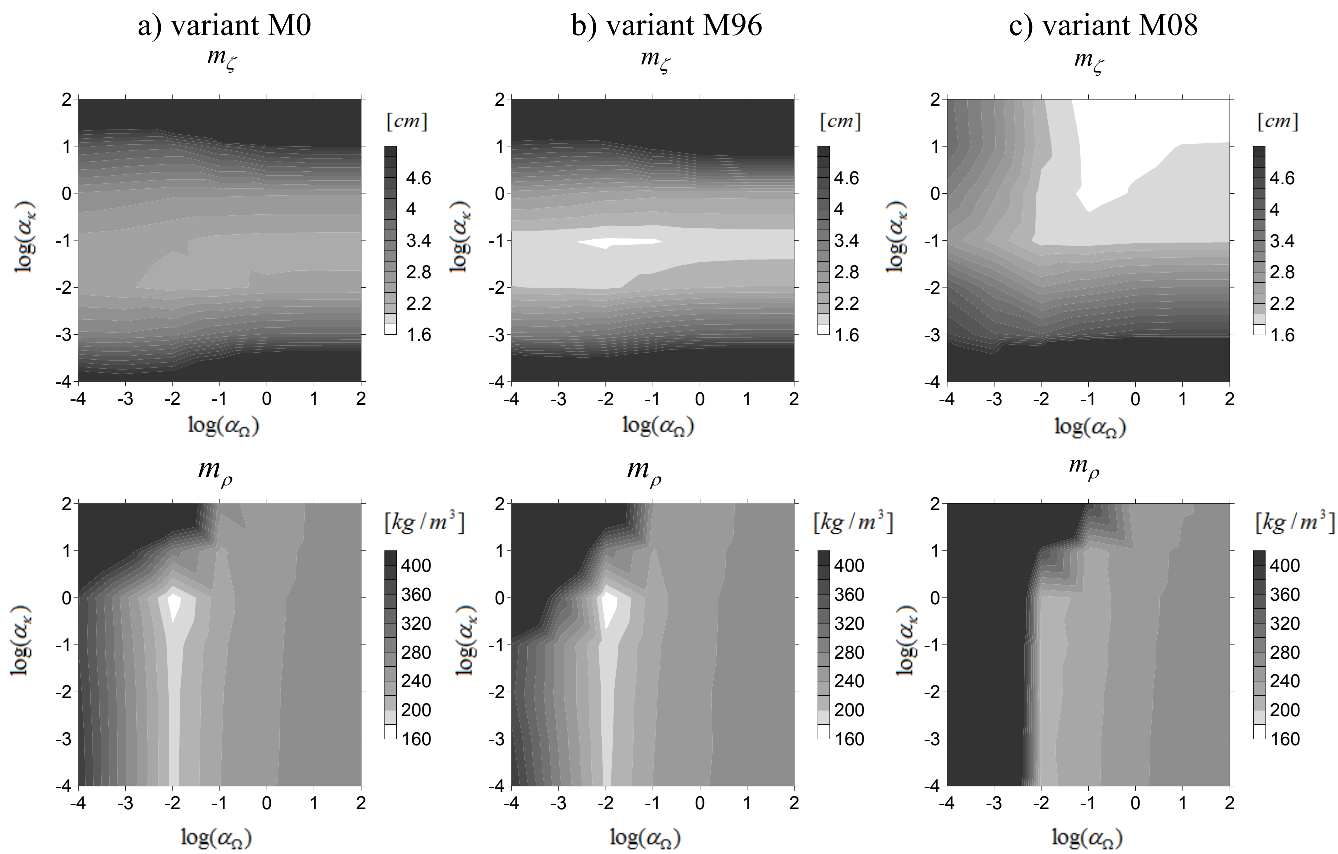


Fig. 8 Parameters m_ζ and m_ρ as a function of coefficients α_Ω and α_κ, for zoning Z2.

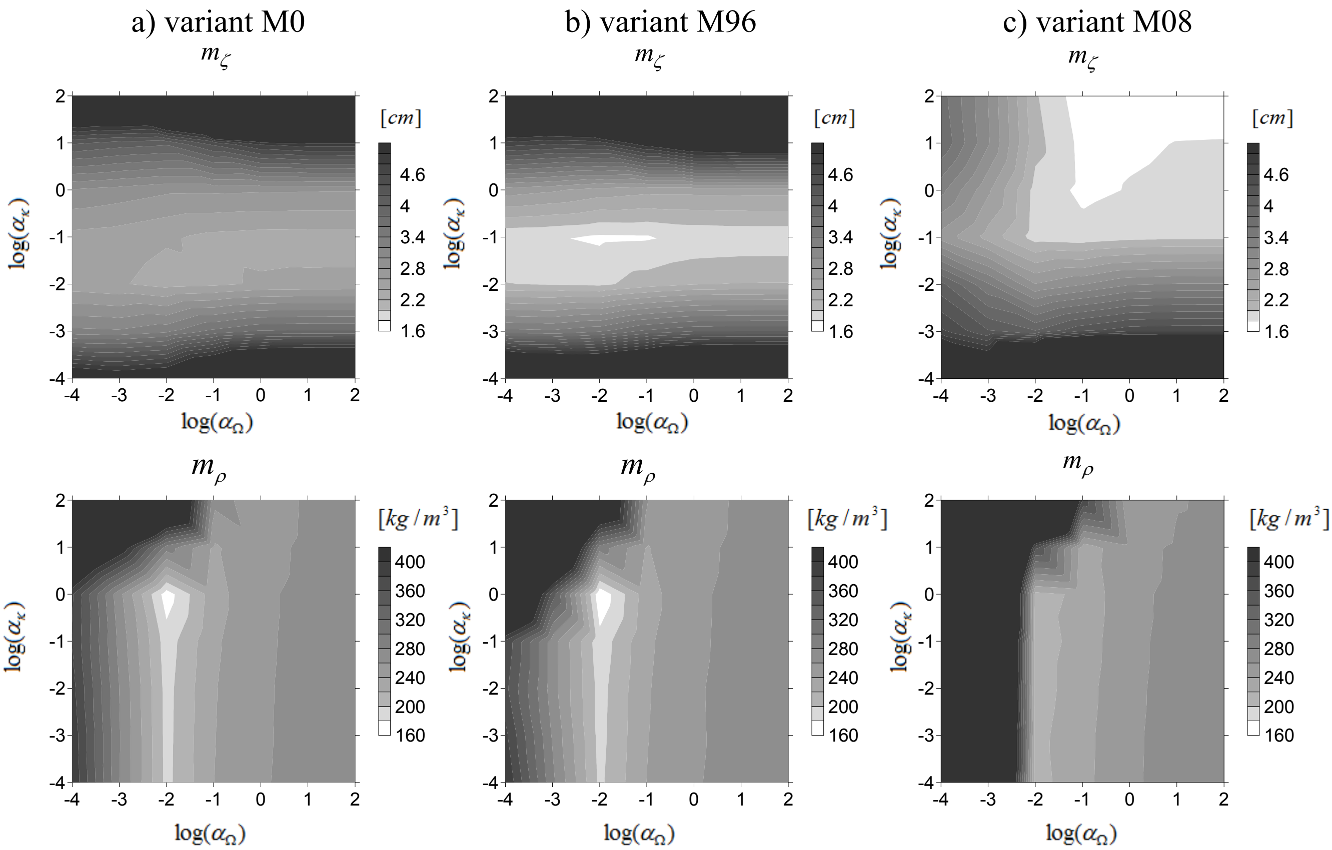


Fig. 9 Parameters m_ζ and m_ρ as a function of coefficients α_Ω and α_κ, for zoning Z3.

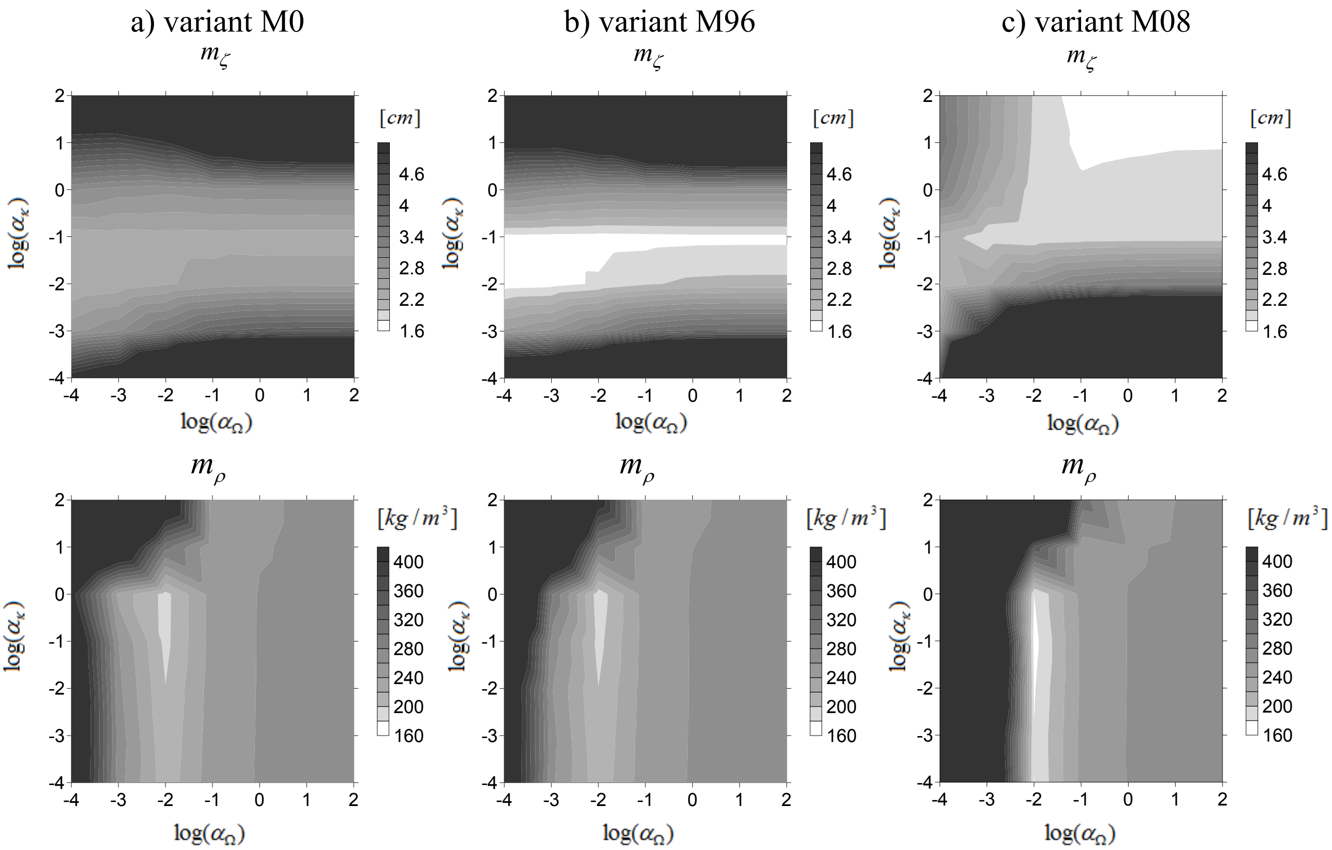


Fig. 10 Parameters m_ζ and m_ρ as a function of coefficients α_Ω and α_κ, for zoning Z4.

3.2. *ESTIMATION OF OPTIMAL VALUES OF COEFFICIENTS α_Ω AND α_κ*

In the preceding paragraph the optimal range of values of coefficient β was defined. For $\beta = 0.0025$ (a value within the optimal range), coefficients α_Ω and α_κ will now be estimated. Figures 7-10 show maps of the errors m_ζ and m_ρ as a function of coefficients α_Ω and α_κ, developed for $\beta = 0.0025$. The maps were developed for each zoning defined in Table 2 and for three variants, i.e. without using any global geopotential model (M0 variant) and using the global models EGM96 (M96 variant) and EGM08 (M08 variant).

To summarize this part of the paper it should be noted that the maps of various variants for all the zoning analyzed are explicitly similar. For maps of the M0 and M96 variants can also be noted that the accuracy of the quasigeoid model depends mainly on coefficient α_κ, relating to the deeper layers of the Earth's crust (below the geoid) and practically does not depend on coefficient α_Ω that relates to the topographic mass (for fixed values of coefficient α_κ, variations in the error m_ζ are very small for the entire test range of coefficient α_Ω). Accuracy of a quasigeoid model is highest for the values $0.1 \le \alpha_\kappa \le 1$ for zoning Z1, and for $0.01 \le \alpha_\kappa \le 0.1$ for zoning Z2, Z3 and Z4.

These relations are somewhat different in the M08 variants. Here, for zoning Z1 the best accuracy of a quasigeoid model is obtained for the coefficients $\alpha_\kappa \ge 1$ and $\alpha_\Omega \ge 0.1$. For zonings Z2, Z3 and Z4 the best coefficients range is a bit wider $\alpha_\kappa \ge 0.1$ and $\alpha_\Omega \ge 0.01$. At this point it should be noted that in the analyses performed the coefficients α_Ω and α_κ for which errors m_ζ are less than or equal $\pm 2cm$ are equally good.

A completely different relationship can be observed in the maps of the accuracy of the density model of topographic mass. The result of modelling of the density of topographic mass is clearly dependent on coefficient α_Ω and to a much lesser extent on coefficient α_κ. The highest accuracy of the density model for almost all variants and all zonings (except variant M08 of zoning Z1) was obtained for coefficient $\alpha_\Omega = 0.01$. For variant M08 of zoning Z1 the optimal value $\alpha_\Omega = 0.1$, should be adopted.

On the basis of the calculations done it is possible to determine optimal values of coefficients α_Ω and α_κ. For zonings Z2, Z3 and Z4 fixed values of these coefficients can be specified as $\alpha_\kappa = 0.1$ and $\alpha_\Omega = 0.01$. These values can be considered optimal also for the M0 and M96 variants of zoning Z1. For this zoning, for M08 variant, due to the accuracy of density of the topographic mass model, the coefficient α_Ω should be equal 0.1. Due to the accuracy of the quasigeoid model, a coefficient $\alpha_\kappa \ge 1$ should be adopted.

Taking this into account, the determined coefficients $\alpha_\kappa = 0.1$ and $\alpha_\Omega = 0.01$ can be considered universal for zoning with a single zone of constant density area which is less than approx. 130 km^2 (zoning Z2, Z3 and Z4). For zoning with a larger single zone of constant density, there are some "shifts" in optimal values of these coefficients.

The found optimal values of the coefficients ($\alpha_\kappa = 0.1$ and $\alpha_\Omega = 0.01$) for variant M08 are essentially boundary values. For quasigeoid modelling slightly different values of the coefficients (e.g. $\alpha_\kappa = 10$ and $\alpha_\Omega = 1$) can be adopted, keeping in mind that the determined density model will not be the best model.

4. CONCLUSIONS

The main objective of this study was to determine the optimal values of the coefficients α_Ω, α_κ and β which are crucial for constructing a model weighting matrix. The importance of global geopotential models for the determined coefficients and the influence of the size of constant density zones on their values were also analyzed. As a result of this research the following conclusions can be formulated:

Because of the accuracy of the density model, the coefficient β should be in the range $0.001 \le \beta \le 0.01$, and the range $0.0025 \le \beta \le 0.005$ should be adopted as the optimal range, regardless of the size of the constant density zone.

The values $\alpha_\kappa = 0.1$ and $\alpha_\Omega = 0.01$ for zones of constant density with area less than approx. 130 km^2 can also be assumed. For larger zones the optimal values of these coefficients are different.

For variants using the global geopotential model EGM08, a slightly better quasigeoid model accuracy was obtained for coefficients slightly larger than indicated above (for example $\alpha_\kappa = 10$ and $\alpha_\Omega = 1$). The use of larger coefficients requires further validation based on more accurate GNSS/leveling data, and unfortunately it is associated with a decrease in accuracy of the topographic mass density model.

At the end it should be noted that the determined optimal values of the coefficients must be verified at other test areas and for more accurate test data.

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