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ORIGINAL PAPER

HARDY FUNCTION MODELING ELEVATION ANOMALY BASED ON TIKHONOV REGULARIZATION

Xiannan HAN

China Railway Design Corporation, Tianjin 300251, China

**Corresponding author's e-mail:* xiannanhan_up@ 163.com

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ABSTRACT

In geoscience research, elevation anomaly modeling plays vital roles in engineering design and construction. In this paper, an improved Hardy function based on Tikhonov regularization is proposed to model elevation anomalies in which the improved model is not constrained by the ill-posed problem induced by that the number of nodes is less than the number of data points, and it has a wider range of use scenarios. In the experimental part, numerical simulation is used to generate simulated data of elevation anomaly, which is divided into training group and checking group, and the hardy functions before and after improvement are used to establish elevation anomaly models, and the performances of the two models are compared. is the results show that the improved Hardy function has better prediction performance with accuracy improvement of 79.75 % due to selecting more nodes and obtaining a more optimal model.

1. INTRODUCTION

It is necessary to model the elevation anomaly to improve the accuracy of projection correction in some engineering applications (e.g. geological research, engineering design and construction and correction of local coordinate system) (Gao and Xu, 2004). In order to improve the accuracy of plane coordinate adjustment calculations, elevation anomaly values within the research area could be obtained by constructing model of elevation anomalies (Liu et al., 2021). Additionally, geodetic heights measured by GNSS receivers and elevation anomalies also could be used to calculated normal heights for aligning to the Chinese height system. Recently, the Hardy function has extensive applications in geoscience. Yang summarized several characteristics of Hardy functions and applied them to rate surface fitting (Yang and Huang, 1990). Wen systematically expounded the theory of hardy functions and inverted the motion state of blocks (Wen and Xu, 2009). Bayer applied hardy functions to calculate the strain parameters of blocks and obtained the strain field of a regional area (Bayer et al., 2006). In addition, many scholars have improved hard functions and broaden their application scope and improved modeling accuracy. Du (2016) replaced the distance-type Kernel function with an exponential-type Kernel function in Hardy functions, and found that the improved method has advantages in describing regional deformations (Du, 2016). Peng introduced Tikhonov regularization into hardy functions to replace the smoothing factor. The improved function is not constrained by the condition where the number of nodes is less than the number of stations, addressing the ill-posed problem of Hardy function. However, this improvement also brings the drawback that the Kernel function cannot be in the form of inverse hyperbolic sine functions (Peng et.al., 2019).

Gathering all previous studies, although hardy function has a wide range of applications in geoscience, scholars continually strive to enhance and innovate the theory. Therefore, this paper presents an improved Hardy model in form of inverse hyperbolic sine function using Tikhonov regularization and utilizes numerical simulation methods to verify the effectiveness of the method finally.

2. HARDY FUNCTION IMPROVED BY TIKHONOV REGULARIZATION

The Hardy function was firstly proposed by Prof. Hardy in 1977, and the key point of this method is that any regular or irregular continuous surface can be represented by the superposition of multiple simple and regular surfaces (Hardy, 1971). The elevation anomaly at any point on a mathematical surface can be expressed as:

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$$\xi(X,Y)\sum_{i=1}^{n}\beta_i K(X,Y,XX_{0,}YY_0) \quad (1)$$

In which, ξ represents the elevation anomaly, X and Y are the geodetic or plane coordinate components of known points, n is the number of selected nodes, XX_{0i} and YY_{0i} represent the geodetic or plane coordinate components of the nodes, β_i stands for the coefficient to be determined, and $K(X,Y,XX_{0i},YY_{0i})$ is a quadratic function of X and Y known as the Kernel function. In geoscience research, symmetric distance-type Kernel functions often yield better results (Hardy, 1978), and the Kernel function can be expressed as:

$$K(X,Y,XX_{0i},YY_{0i},\delta)=(dis^a+\delta^2)^b \quad (2)$$

Where dis denotes the distance between the selected position and the node, with its value being $dis=\left((X_i-XX_{0i})^2+\left(Y_j-YY_{0j}\right)^2\right)^{\frac{1}{2}}$. a and b are two adjustable variables, and different values correspond different Kernel function forms (Du, 2016). When a is 2 and b is -1/2, the Kernel function is the inverse hyperbolic sine function. When a is 2 and b is 1/2, the Kernel function is defined as hyperbolic sine function. While a is 3 and b is 1, the Kernel function is a cubic surface function. δ^2 is the smoothing factor, used to adjust the shape of the Kernel function, and the appropriate value can improve the stabilities and accuracies of the model. Typically, δ^2 is chosen as a random number between 1 and 10 (Hu and Sun, 2003). Assuming the number of known elevation anomaly points is m and the number of selected nodes is n, then the equation is constructed:

$$\begin{bmatrix}\xi(L_1,B_1)\\ \xi(L_2,B_2)\\ \vdots\\ \xi(L_m,B_m)\end{bmatrix}= \\ =\begin{bmatrix}K(L_1,B_1,L_{01},B_{01}) & K(L_1,B_1,L_{02},B_{02}) & \cdots & K(L_1,B_1,L_{0n},B_{0n})\\ K(L_2,B_2,L_{01},B_{01}) & K(L_2,B_2,L_{02},B_{02}) & \ldots & K(L_2,B_2,L_{0n},B_{0n})\\ \vdots & \vdots & \ddots & \vdots\\ K(L_m,B_m,L_{01},B_{01}) & K(L_m,B_m,L_{02},B_{02}) & \ldots & K(L_m,B_m,L_{0n},B_{0n})\end{bmatrix}\boldsymbol{\beta} \quad (3)$$

The vector-matrix form is given by:

$$\theta=K\beta \quad (4)$$

In the equation, K represents the coefficient matrix, and β signifies the parameter vector.

When establishing a velocity field model using hardy function, it is required that the number of stations m is greater than the number of nodes n. When the number of nodes exceeds the number of stations, i.e., the unknown parameters exceed the number of equations, it leads to an unsolvable parameter situation. However, when the number of stations is relatively small, the available nodes are limited, resulting in a model that lacks good applicabilities. Therefore, L2 regularization is introduced to improve the classical Hardy function, which removes the constraint that the number of Kernel function n cannot exceed the number of stations m. When $m < n$, due to the introduction of the regularization term, the matrix remains invertible, and the model still has a unique solution. The L2 regularization improves the hardy function by addressing ill-posed problems while preventing overfitting, thus enhancing the generalization capabilities of the model.

Introducing Tikhonov regularization matrix and imposing the condition of minimizing the norm of the parameter vector, the coefficient solution can be obtained according to the following equation:

$$\hat{\beta}=argmin[(\theta-K\beta)^TQ^{-1}(\theta-K\beta)+u\|\beta\|_2^2] \quad (5)$$

The formula for the analytical solution is:

$$\hat{\beta}=argmin(K^TPK+uD)^{-1}K^TP\theta \quad (6)$$

In which, $\boldsymbol{K}$ is the Kernel function matrix (Jin,2005), $\boldsymbol{P}$ represents the weights of the observed values. In this example, randomly chosen uniformly distributed grid points are used as nodes, thus the weights are all set to 1 (Peng et al., 2019), i.e., $P=Q^{-1}$, where $\boldsymbol{P}$ signifies the identity matrix. u is the regularization parameter, and $\boldsymbol{D}$ is the positive definite regularization matrix, typically an identity matrix. Elevation anomaly value at any point $\left(X_q,Y_q\right)$ can be obtained through Equation (1), resulting in an elevation anomaly distribution map (Tang et al., 2018; Wang et al., 2016).

3. SEVERAL IMPORTANT PARAMETERS IN THE IMPROVED HARDY FUNCTION

There are three main issues to consider with Hardy functions:

1. Kernel function, which is different for different data, and the previous studies have also drawn different conclusions. When fitting the crustal vertical movement using the Hardy function, it is considered that the fitting result of inverse hyperboloid function is the best, followed by the positive hyperboloid function (Yang and Huang, 1990). However, it is considered that using positive hyperboloid function has the best fitting result in establishment of China crustal horizontal movement velocity field (Liu et al., 2002). In this paper, the experimental results show that the optimal Kernel function is inverse hyperboloid (Han et al., 2021).
2. Smoothing factor, which can slightly change the shape of the Kernel function curve, and has the effect on smoothing the curve (Fang and Huang, 2022). Being in the manipulation of modeling, the normal matrix is required to be non-singular. So that the Kernel curve will be distorted to a certain extent with the increase of smoothing factor, and the relationship between Kernel function and the distance is non-linear (Peng, 2019). The larger the value of smoothing factor, the greater the distortion of Kernel curve, the worse the linear correlation with distance, and the higher the stability of the model (Zhang and Lv, 2017). But at the same time, the fitting error (not necessarily the interpolation/prediction error) of the model

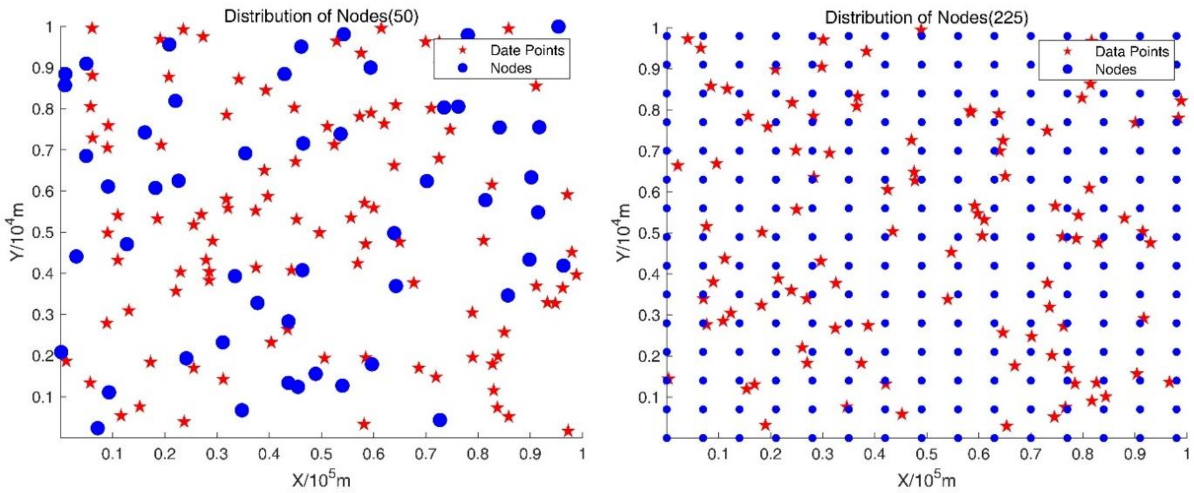

Fig. 1 Distribution of stations and nodes.

will also be increased. Empirically, the most appropriate value of smoothing factor is set between 1-10 and we set this value as 1 (The value is 1-10 gives similar experimental results) (Zhang et al., 2010).

3. Nodes, which has a significant influence on fitting and prediction. There has been systematic selection method on this: The interpolation accuracy of the hardy function model increases with the increase of the number of nodes (i.e. the spacing of nodes decreases) (Huang et al., 1993), and all sites and regularly distributed grids are used as nodes in this article (Yang, 1990). The observations uniformly distributed are selected as nodes in constructing China crustal horizontal movement velocity field model (Liu et al., 2002). All observations are selected as nodes when using Tikhonov regularization to improve the hardy function fitting performance (Peng, 2019). In elevation anomalies modeling, the number of nodes should be determined (Huang et al., 1993). On one side, if the number n of Kernel function is greater than the number m of station, the solution could not be found. On the other side, the number of nodes is too insufficient to represent the detailed pattern of the elevation anomalies. In this paper, two node selection strategies were determined, randomly and evenly distributed nodes.

4. SIMULATION

Sakamoto's simplified model was used to represent the real model and generate simulated data in this paper (Burnham and Anderson, 2002; Xu et al., 2018). The Sakamoto model is as follows:

$$y = exp[(x - 0.3)^2] - 1 + \varepsilon \quad (7)$$

In the equation, $x \sim U\,(0,1)$ represents a random variable following a uniform distribution on the interval (0,1), and ε represents the observation error of y, following a normal distribution with a mean of μ and a standard deviation of δ, i.e.$\varepsilon \sim N\,(\mu, \delta^2)$. The elevation anomaly $\xi(X, Y)$ is assumed to be a function of coordinates X and Y. This function captures the relationship between the elevation anomaly and the coordinates in the Sakamoto model.

$$\xi = exp[(x - 0.3)^2 + (y - 0.5)^2] - 1 + \varepsilon \quad (8)$$

80 elevation anomaly values for the training set are generated by the model described above. Additionally, 20 elevation anomaly values are generated for prediction validation. These 100 generated data points are randomly distributed within a 100 km × 100 km area. The observation error ε follows a normal distribution with a mean 0 and a standard deviation 0.01, and the coordinates range from 0 to 1, with a unit of 10^5 meters. The prediction abilities of the classical Hardy function and the improved Hardy function are compared under two node selection strategies. The data and node distribution of two strategies is shown in Figure 1.

The experiment suggests that the Kernel function of positive hyperbolic sine-shaped form performs better in prediction compared to the in the forms of inverse hyperbolic sine and cubic surface-shaped. The choice of the smoothing factor in the range of 0.01 to 1 has a minimal impact on the results. Therefore, this paper presents a comparison between the two types of hardy functions with respects to different node distribution, both based on the positive hyperbolic sine-shaped form with a smoothing factor of 0.1.

In experiment 1, 50 randomly distributed nodes in scheme 1 were selected, 80 elevation anomaly data generated by Sakamoto model were used to establish models using classical hardy function and hardy function improved by Tikhonov regularization, and then the calculating 20 values at checking points. Under this scheme, the prediction results of the two models are shown in Figure 2.

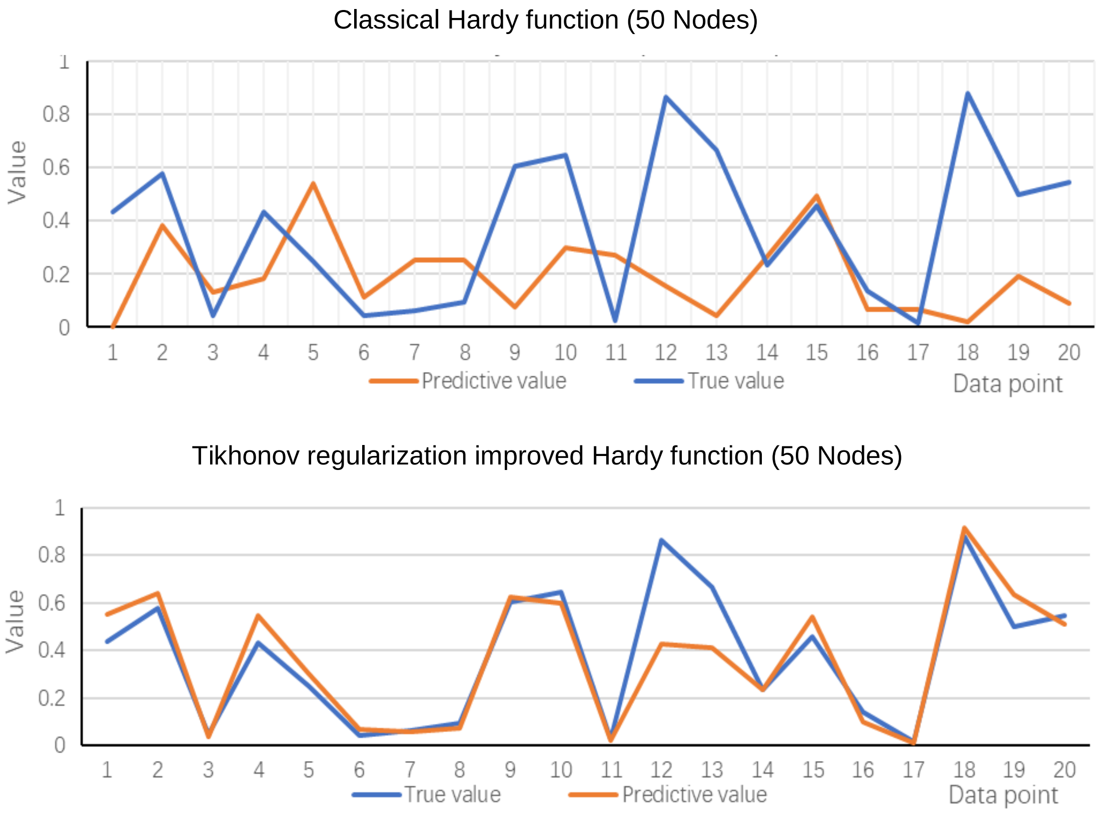

Fig. 2 Prediction results of Hardy functions before and after of Tikhonov regularization improvement.

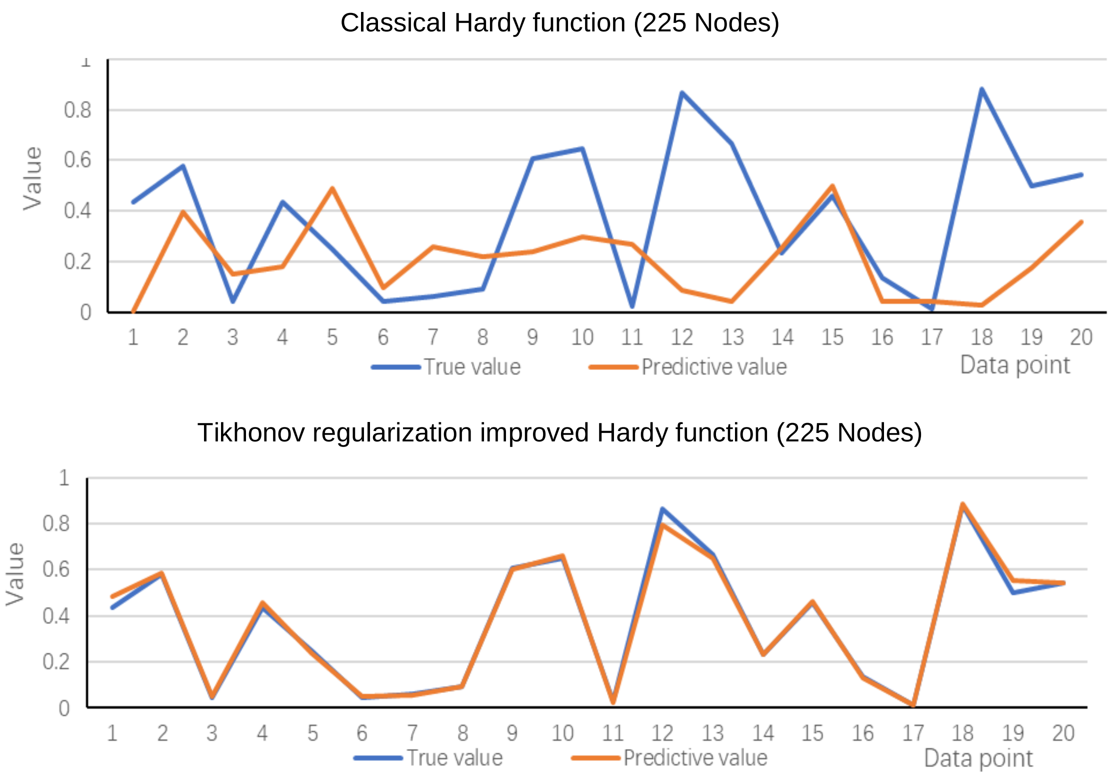

Fig. 3 Prediction results of Hardy functions before and after Tikhonov regularization improvement.

In experiment 2,225 uniformly distributed nodes in scheme 2 were selected, and the same 80 abnormal elevation data in scheme 1 were used to establish models using classical Hardy function and the improved hardy function respectively, and then the values at 20 points were predicted. Under this scheme, the prediction results of the two models are shown in Figure 3.

After the calculation and comparison of the two groups of experiments, we can get the following results. In the strategy of 50 randomly distributed nodes, the root means square error (RMSE) of the predicted value of the classical hardy function is 0.3774, while the RMSE of Hardy function improved by Tikhonov regularization is significantly decreased, which is 0.1264. After adding nodes, the RMSE of classical hardy function is 0.3596, and the RMSE of the improved hardy function is 0.0242.

5. CONCLUSION

In the simulation experiments, it can be clearly found that under the 50 randomly distributed node strategies, the improved Hardy function by Tikhonov regularization has better prediction performance, and its accuracy is 66.5 % higher than that of the classical one Similarly, when selecting the uniformly distributed node strategies, the improved model has better prediction accuracy of 93 %. In addition, it can be intuitively seen that when the nodes number increases, the prediction accuracy of the classical Hardy function increases by 4.7 %, while the accuracy of the improved Hardy function increases by 80.8 %. Therefore, it can be concluded that under the condition that with the same number of nodes, the improved Hardy function has better prediction performances than the classical one with higher prediction accuracy of 79.75 % on average under the two node selection strategies. When the number of nodes increases, the improved Hardy function ensures better performance, which is of great significance for predicting elevation anomalies. the application of Hardy function in elevation anomalies is explored preliminary in this paper, providing insights for improving engineering design and construction accuracy.

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