

# A MODEL OF SIMULTANEOUS DISTRIBUTION OF HUMIDITY AND TEMPERATURE IN POROUS MATERIALS

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*A brief derivation of a so-called Kiessl model for the humidity behaviour of materials with pore structure in dimension  $n = 1, 2, 3$  is presented. In this way, both the essential processes which this model does describe are pointed-out and the possible meanings of the related numerical modelling are indicated.*

## INTRODUCTION

It is typical for the building materials of common use that a large part of their volume, up to 60 % in the case of cellular concrete, is filled in by pores with extremely diverse size and shape. This is the reason why the building materials are able to absorb water both in the form of a liquid and of a gas, save it in its pores, transport it and discharge it. This so-called *humidity behaviour* of building materials has been evaluated by a methodology of Glaser, originated in [4], according to the norms of most European countries. But today we have a large amount of evidence showing that conclusions, obtained by this methodology, differ from reality essentially in many important cases. This concerns for example constructions composed of materials with extensive differences in their humidity behaviour. The number of such constructions corresponds to the growing interest in building objects with small consumption of heat. One of the reasons why the Glaser model is not accurate enough is that it does not take into account strong connections between the humidity behaviour and other simultaneous processes in building constructions. Especially, there is a strong relation between humidity and temperature. Humidity has an essential influence on the diffusion of gases from the environment into the material. For example the concentration of carbon dioxide in a building material determines its durability. Presence of a high amount of humidity leads to a change of volume by discharging certain components of the material and, consequently, to the distress of stability of the whole object.

A theory necessary for the numerical modelling of humidity behaviour started to develop much later than for other processes like deformation or distribution of temperature. Even the physical basis for this theory is not complete yet. On the other hand, there is a big demand for any kind of prediction of humidity behaviour especially in parts of constructions where humidity in the liquid phase appears. For special concrete problems of this kind, the use of empirical models is typical.

Physics of building materials studies the ability of materials to minimize the negative influence of humidity intensively for a number of decades. But the search after essential notions, derivation of governing rules and formulation of criteria for quality evaluation concentrates mostly on some special features of the system water - building material only.

As many non-successful comparisons with measured data illustrate, no simple model can produce any reasonably accurate estimate of the real humidity behaviour. The reason is an inherent complexity of this process, which is influenced by the following factors (a)-(d) essentially.

- (a) The shape and size of pores.
- (b) The presence of humidity in pores both in the liquid and in the gas phase at the same time.
- (c) An interaction of transport of various kinds: capillary convection, diffusion, surface diffusion, effusion.
- (d) The temperature which activates various kinds of transport with various intensity.

Conversely, the process of distribution of temperature depends on humidity to a large extent.

Every more exact model of humidity behaviour has to be formulated in terms of quantities of potential type which encode the relative part of water in each phase within humidity. Such potential quantities have been described earlier by Bogoslovskij [1], Krischer [7], Lykow [9] and de Vries [10]. Under natural conditions, the interaction between humidity and temperature cannot be neglected.

Like the most mathematical models of humidity behaviour, the Kiessl model (KM in what follows) has been formulated for the one-dimensional case only. It takes into account all the factors (a) - (d) and, besides temperature  $\tau = \tau(x, t)$  ( $^{\circ}\text{C}$ ), it works with a humidity potential  $\Phi = \Phi(x, t)$  (-) defined in an original way. In agreement with the general theory of Onsager it expresses the global diffusive transport as a superposition of partial transports proportional to gradients of several quantities.

#### DERIVATION OF THE KIESSL MODEL

The kernel of the KM consists of two transient partial differential equations. It is apparent from the following derivation of these equations (in a region  $\Omega \subset \mathbf{R}^n$ ,  $n = 1, 2, 3$  and in time-interval  $(0, T)$ ), which processes does the model take into account.

We denote by

$M$	( $\text{kg m}^{-3}$ )	the weight of humidity in $1 \text{ m}^3$ of the porous material,
$H$	( $\text{J m}^{-3}$ )	the amount of heat in $1 \text{ m}^3$ ,
$W$	( $\text{J m}^{-3} \text{ s}^{-1}$ )	the intensity of sources of heat,
$I$	( $\text{kg m}^{-3} \text{ s}^{-1}$ )	the intensity of condensation of vapour,
$\tau$	( $^{\circ}\text{C}$ )	the temperature,
$u$	(-)	the volume taken by water and ice in $1 \text{ m}^3$ as if all this humidity were in the liquid phase,
$\varphi$	(-)	the relative humidity of the air in pores,
$\rho_w$	( $\text{kg m}^{-3}$ )	the density of water,
$D_u \nabla u$	( $\text{kg m}^{-2} \text{ s}^{-1}$ )	the intensity of capillar flow of humidity,
$D_\varphi \nabla \varphi$	( $\text{kg m}^{-2} \text{ s}^{-1}$ )	the intensity of flow of vapour at the gradient of relative humidity,
$D_\tau \nabla \tau$	( $\text{kg m}^{-2} \text{ s}^{-1}$ )	the intensity of flow of humidity at the gradient of temperature,
$\lambda \nabla \tau$	( $\text{J m}^{-2} \text{ s}^{-1}$ )	the intensity of flow of heat,

take arbitrary cube  $Q \subset \Omega$  and interval  $(t_0, t_0 + \Delta t) \subset (0, T)$ . The first identity (1) expresses the law of preservation of mass: The accession of humidity in  $Q$  from  $t_0$  to  $t_0 + \Delta t$  (the left-hand side) equals to the amount of humidity which penetrates into  $Q$  through the boundary  $\partial Q$  (the

right-hand side). The second identity (2) encodes the law of preservation of (thermal) energy: The accession of heat (the left-hand side) equals to the heat which penetrates into  $Q$  through the boundary  $\partial Q$  and to the heat created in the sources inside of  $Q$  (the right-hand side). The third identity (3) says that the accession of  $\rho_w u$  (the left-hand side) can appear either by the capillar flow through  $\partial Q$  or by the condensation of vapour (the right-hand side).

$$\int_Q [M(t_0 + \Delta t) - M(t_0)] d\Omega = \int_{t_0}^{t_0 + \Delta t} \int_{\partial Q} \left( D_u \frac{\partial u}{\partial n} + D_\varphi \frac{\partial \varphi}{\partial n} + D_\tau \frac{\partial \tau}{\partial n} \right) ds dt \quad (1)$$

$$\int_Q [H(t_0 + \Delta t) - H(t_0)] d\Omega = \int_{t_0}^{t_0 + \Delta t} \int_{\partial Q} \lambda \frac{\partial \tau}{\partial n} ds dt + \int_{t_0}^{t_0 + \Delta t} \int_Q W d\Omega dt \quad (2)$$

$$\rho_w \int_Q [u(t_0 + \Delta t) - u(t_0)] d\Omega = \int_{t_0}^{t_0 + \Delta t} \int_{\partial Q} D_u \frac{\partial u}{\partial n} ds dt + \int_{t_0}^{t_0 + \Delta t} \int_Q I d\Omega dt \quad (3)$$

By an obvious modification of the left-hand sides and by means of the Green theorem (see [2], p.89) on the right-hand sides, we obtain the following identities (4) - (6) from (1) - (3).

$$\int_{t_0}^{t_0 + \Delta t} \int_Q \left[ \frac{\partial M}{\partial t} - \nabla (D_u \nabla u + D_\varphi \nabla \varphi + D_\tau \nabla \tau) \right] d\Omega dt = 0 \quad (4)$$

$$\int_{t_0}^{t_0 + \Delta t} \int_Q \left[ \frac{\partial H}{\partial t} - \nabla (\lambda \nabla \tau) - W \right] d\Omega dt = 0 \quad (5)$$

$$\int_{t_0}^{t_0 + \Delta t} \int_Q \left[ \rho_w \frac{\partial u}{\partial t} - \nabla (D_u \nabla u) - I \right] d\Omega dt = 0 \quad (6)$$

We multiply the equations (4) - (6) by  $1/\Delta t$ , pass to the limit for  $\Delta t \rightarrow 0$  on the assumption that the integrands in (4) - (6) are continuous (see [8], p.61) and obtain the following differential equations (7) - (9) by means of the Lebesgue theorem (see [5], p.33).

$$\frac{\partial M}{\partial t} - \nabla (D_u \nabla u + D_\varphi \nabla \varphi + D_\tau \nabla \tau) = 0 \quad (7)$$

$$\frac{\partial H}{\partial t} - \nabla(\lambda \nabla \tau) - W = 0 \quad (8)$$

$$\rho_w \frac{\partial u}{\partial t} - \nabla(D_u \nabla u) - I = 0 \quad (9)$$

In the KM, the unknown functions are the *humidity potential*  $\Phi = \Phi(x, t)$  (-) and the *temperature*  $\tau = \tau(x, t)$  (°C). In order to express the equations (7), (8) in terms of  $\tau$ ,  $\Phi$  and of given data, we use the following known constants and functions:

$\rho_w$	(kg m <sup>-3</sup> )	density of water,
$c_w$	(J kg <sup>-1</sup> K <sup>-1</sup> )	specific heat of water,
$\rho_E$	(kg m <sup>-3</sup> )	density of ice,
$c_E$	(J kg <sup>-1</sup> K <sup>-1</sup> )	specific heat of ice,
$\rho_M$	(kg m <sup>-3</sup> )	density of porous material,
$c_M$	(kg <sup>-1</sup> K <sup>-1</sup> )	specific heat of porous material,
$P$	(-)	volume of pores in 1 m <sup>3</sup> of porous material,
$\chi$	(-)	part of the mass of $u$ created by water,
$c_s$	(kg m <sup>-3</sup> )	partial pressure of saturated vapour in the air,
$L_{1,3}$	(J kg <sup>-1</sup> )	specific heat of sublimation of ice,
$L_{2,3}$	(J kg <sup>-1</sup> )	specific heat of evaporation of water and
$L_{1,2}$	(J kg <sup>-1</sup> )	specific latent heat of ice.

The function  $c_s$  can be approximated by the function:

$$\frac{A\tau+B}{e^{\tau+C}},$$

$$R(\tau+273.15)$$

where  $R = 461.9$  (J kg<sup>-1</sup> K<sup>-1</sup>) and values  $A, B, C$  are the following:

	$\tau < 0$	$\tau \geq 0$
$A$	28.9205	23.589991
$B$	1751.21042	1513.86688
$C$	273	236

One can easily see that

$$u\chi + u(1 - \chi) \frac{\rho_w}{\rho_E}$$

and

$$P - u\chi - u(1 - \chi) \frac{\rho_w}{\rho_E}$$

is that part of the volume of pores in 1 m<sup>3</sup> of porous material which is filled in by water or ice and by air

respectively. Moreover,  $\varphi c_s(\tau)$  is the mass of vapour in 1 m<sup>3</sup> of air at the temperature  $\tau$  and relative humidity  $\varphi$ . Hence, we have

$$M(t) = \rho_w u + \varphi c_s(\tau) [P - u\chi - u(1 - \chi) \rho_w / \rho_E]$$

The diffusion coefficients  $D_u, D_\varphi$  and  $D_\tau$  have been expressed in the following forms on the basis of experiments in Kiessl [6]:

$$D_u = \rho_w \varepsilon_\kappa(\tau) \kappa(u),$$

$$D_\varphi = \rho_w \varepsilon_\varphi(\tau) k_{u\varphi}(u),$$

$$D_\tau = \rho_w \varepsilon_\tau(\tau) k_{d\tau}(u).$$

The functions  $\kappa, k_{u\varphi}, k_{d\tau}, \varepsilon_\kappa, \varepsilon_\varphi, \varepsilon_\tau$  and  $\lambda$  depend on the concrete porous material. Graphs of typical examples can be seen in figures 1, 2 and 3.

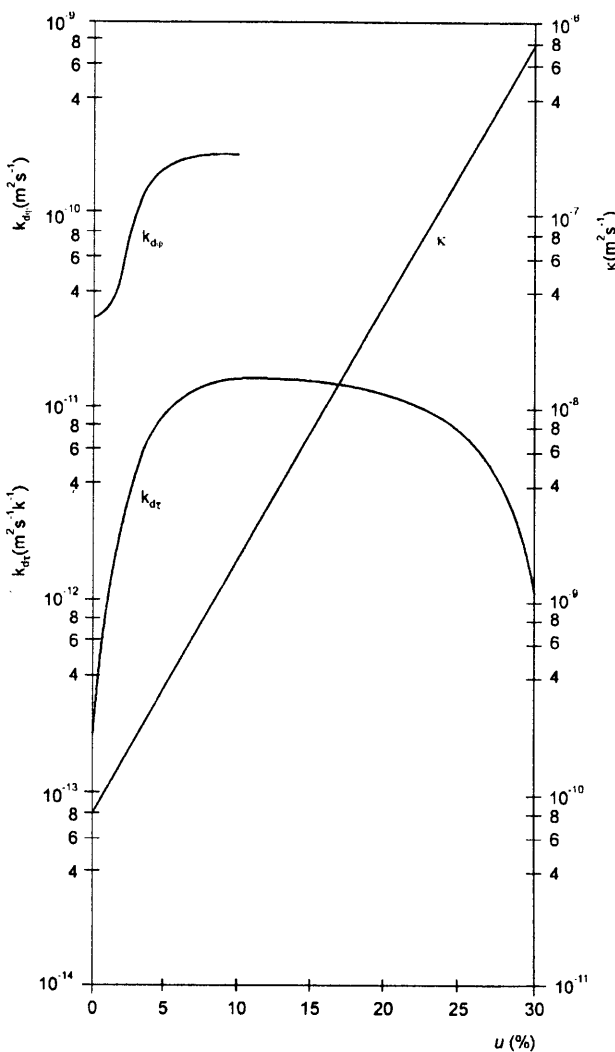


Figure 1. Graphs of the function  $\kappa, k_{u\varphi}, k_{d\tau}$  (brick body).

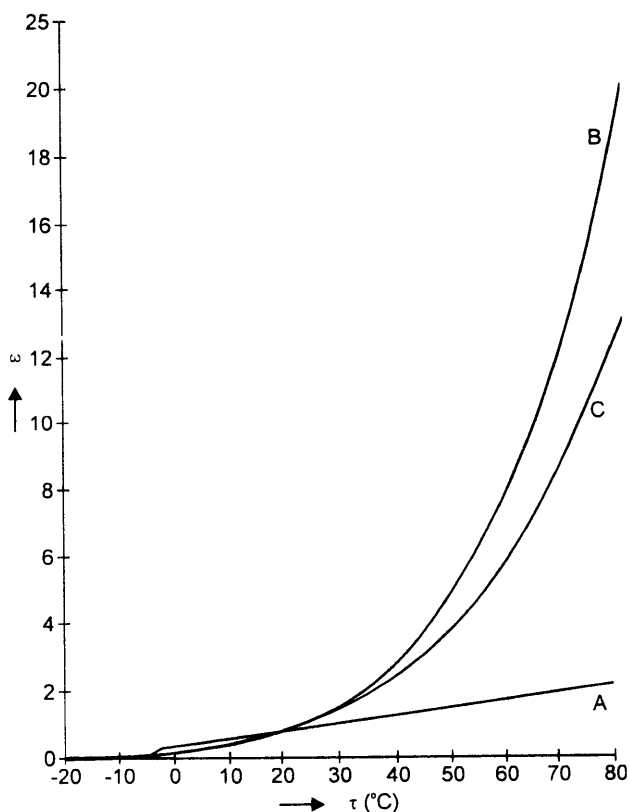


Figure 2. Graphs of the function  $\epsilon_{\kappa}$ ,  $\epsilon_{\varphi}$ , and  $\epsilon_{\tau}$  (brick body).  
A - graph of  $\epsilon_{\kappa}$ , B - graph of  $\epsilon_{\varphi}$ , C - graph of  $\epsilon_{\tau}$

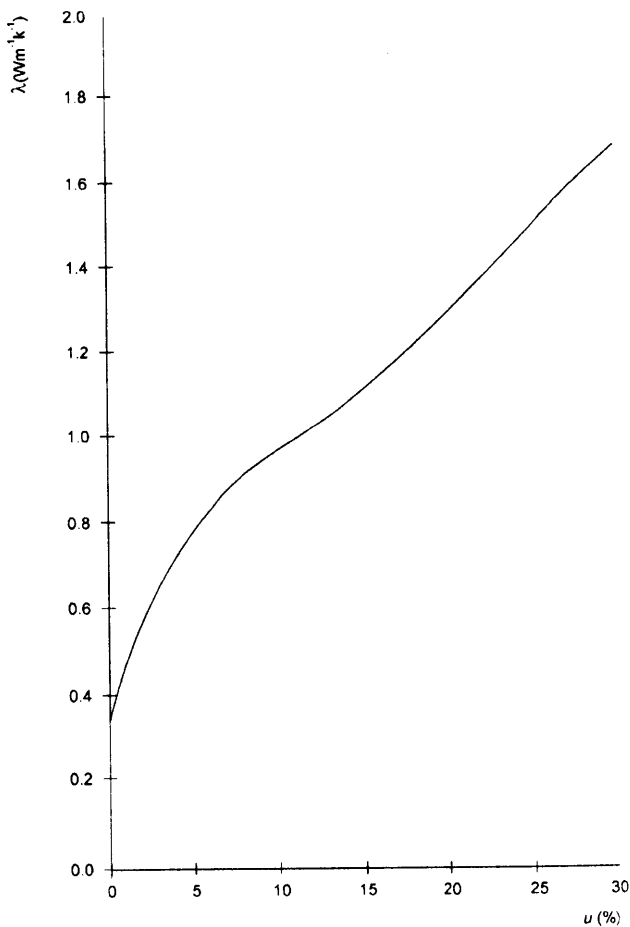


Figure 3. Graph of the function  $\lambda$  (brick body).

Substituting the above-described expressions of the functions  $M$ ,  $D_u$ ,  $D_\varphi$  and  $D_\tau$  into (7), we obtain the following *humidity equation*

$$\begin{aligned} \frac{\partial}{\partial t} \left( \rho_w u + \left[ P - u\chi - u(1-\chi) \frac{\rho_w}{\rho_E} \right] \varphi c_s(\tau) \right) = \\ = \rho_w \nabla [\epsilon_{\kappa}(\tau) \kappa(u) \nabla u + \epsilon_{\varphi}(\tau) k_{u\varphi}(u) \nabla \varphi + \\ + \epsilon_{\tau}(\tau) k_{u\tau}(u) \nabla \tau] . \end{aligned} \quad (10)$$

The heat in  $1 \text{ m}^3$  can be expressed in the form

$$H = \rho_M c_M \tau + \rho_w c_w u \chi \tau + \rho_w c_E u(1-\chi) \tau + \rho_w u \chi L_{1,2}$$

as a sum of the heat in the porous material, in the water in pores, in the ice in pores and of the specific latent heat of the water in pores. If we assume that the process of condensation changes vapour to water and ice in the relation prescribed by the value of the function  $\chi$  then we obtain the intensity of sources of heat in the form

$$W = (\chi L_{2,3} + (1-\chi) L_{1,3}) I = L_{\dots,3} I =$$

$$= L_{\dots,3} \left[ \rho_w \frac{\partial u}{\partial t} - \nabla (D_u \nabla u) \right]$$

according to (9). Now, if we insert the above expressions of  $H$  and  $W$  into (8) then we obtain the *temperature equation*

$$\begin{aligned} \frac{\partial}{\partial t} [\rho_M c_M \tau + \rho_w c_w u \chi \tau + \rho_w c_E u(1-\chi) \tau + \\ + \rho_w u \chi L_{1,2}] - \nabla (\lambda \nabla \tau) = \\ = L_{\dots,3} \left[ \rho_w \frac{\partial u}{\partial t} - \nabla (\rho_w \epsilon_{\kappa}(\tau) \kappa(u) \nabla u) \right] \end{aligned} \quad (11)$$

The remaining step of the derivation of the KM consists in introducing the humidity potential  $\Phi$  [-]. K. Kiessl defined this function by

$$\Phi = \varphi \text{ for } \varphi \leq 0.9 \text{ and } \Phi = 1.7 + 0.1 \log r \text{ for } \varphi > 0.9 .$$

Here  $r$  is a mean value of the diameter of pores in the porous material. The dependence of humidity  $u$  on the humidity potential is given by a so-called sorption isotherm  $f$ . In the cases  $\varphi \leq 0.9$ ,  $f(\Phi)$  is that part of  $1 \text{ m}^3$  of the porous material which is filled in by water under given relative humidity. An example is presented in figure 4.

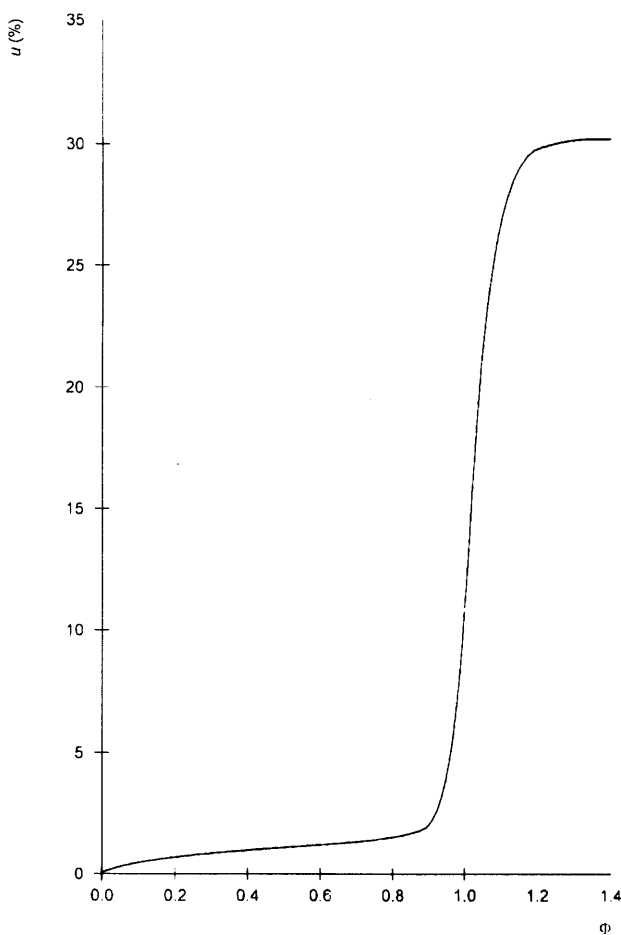


Figure 4. Graph of the function  $\Phi$  (brick body).

If we introduce the notation

$$c_{wE} = c_w - c_E, \quad g = \varphi \circ f, \quad \Lambda = \lambda(\tau, f(\Phi)), \quad \Theta = \kappa \circ f, \\ K_{d\varphi} = k_{d\varphi} \circ f, \quad K_{d\tau} = k_{d\tau} \circ f,$$

then we can reformulate the equations (10), (11) in the following final forms (12), (13):

The equation of humidity potential

$$\frac{\partial}{\partial t} \left( \rho_w f(\Phi) + \left[ P - f(\Phi) \left( \chi + (1-\chi) \frac{\rho_w}{\rho_E} \right) \right] g(\Phi) c_s(\tau) \right) = \\ = \rho_w \nabla [\varepsilon_\kappa(\tau) \Theta(\Phi) f'(\Phi) \nabla \Phi + \varepsilon_\varphi(\tau) K_{d\varphi}(\Phi) g'(\Phi) \nabla \Phi + \\ + \varepsilon_\tau(\tau) K_{d\tau}(\Phi) \nabla \tau] \quad (12)$$

and the temperature equation

$$\frac{\partial}{\partial t} [\rho_M c_M \tau + \rho_w \tau f(\Phi) (c_E + c_{wE} \chi(\tau)) + \\ + \rho_w L_{1,2} f(\Phi) \chi(\tau)] = \\ = \nabla (\Lambda(\tau, \Phi) \nabla \tau) + \\ + L_{3,3}(\tau) \rho_w \left[ f'(\Phi) \frac{\partial \Phi}{\partial t} - \nabla (\varepsilon_\kappa(\tau) \Theta(\Phi) f'(\Phi) \nabla \Phi) \right]. \quad (13)$$

The equations (12), (13) together with the boundary conditions

$$(\varepsilon_\kappa(\tau) \Theta(\Phi) f'(\Phi) + \varepsilon_\varphi(\tau) K_{d\varphi}(\Phi) g'(\Phi)) \frac{\partial \Phi}{\partial n} + \\ + \varepsilon_\tau(\tau) K_{d\tau}(\Phi) \frac{\partial \tau}{\partial n} = g_1(x, t, \Phi, \tau) \\ - L_{3,3}(\tau) \rho_w \varepsilon_\kappa(\tau) \Theta(\Phi) f'(\Phi) \frac{\partial \Phi}{\partial n} + \Lambda(\tau, \Phi) \frac{\partial \tau}{\partial n} = \\ = g_2(x, t, \Phi, \tau) \quad (14)$$

and the initial conditions

$$\Phi(x, 0) = \Phi_0(x), \quad \tau(x, 0) = \tau_0(x) \quad (15)$$

form the so-called *Kiessl model*.

Under certain natural assumptions on coefficients which are satisfied for all known materials, the KM is a second-order strongly parabolic quasilinear system of "reaction - diffusion" type. By means of known a priori estimates for the solutions of linear parabolic systems and using a certain fixed point theorem (see [3]) we have proved the existence of smooth exact solution on a finite time-interval.

We are searching after a numerical method for an accurate and stable approximation of the solution of KM in the one- and two- dimensional cases for constructions with generally composite materials. We hope that we obtain a reliable computer program which will give us more exact informations about the humidity behaviour of constructions. But even accurate numerical solutions of the KM will not describe the reality in all cases. For example, the KM is not able to consider changes of the properties of building materials in time or the influence of unpredictable flow of air in the hollows of constructions. We hope that this program will be useful in critical cases like the humidity behaviour of flat roofs or that of mantles of cooling towers.

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MODEL  
SOUČASNÉHO ŠÍŘENÍ VLHKOSTI A TEPLA  
V PORÉZNÍCH MATERIÁLECH

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Obsahem příspěvku je exaktní odvození tzv. Kiesslova modelu v jedno, dvou i trojrozměrném případě. Jádrem tohoto modelu je systém dvou parciálních diferenciálních rovnic, popisující šíření vlhkosti a tepla v materiálech s pórovitou strukturou. Z odvození jsou patrné předpoklady, za nichž Kiesslův model tento proces popisuje dostatečně výstižně. Poznatky autorů o struktuře a řešitelnosti tohoto modelu i cíle jejich výzkumu jsou stručně okomentovány. V práci jsou zmíněny i některé konkrétní důležité případy možného využití Kiesslova modelu.